MA 485-1C (Probability), Dr. Chernov | Midterm test #2

5 problems, each is worth 20 points. Show your work. | Mon, Oct 31, 2011

1. Find the following probabilities for two normal random variables: $Z = \mathcal{N}(0,1)$ and $X = \mathcal{N}(-1,16)$.

(a) $P(Z > -2.17) =$ Answer: $1 - \Phi(-2.17) = 0.9850$

(b) $P(|X| < 3.2) =$ Answer: $F_X(3.2) - F_X(-3.2) = \Phi\left(\frac{3.2+1}{4}\right) - \Phi\left(\frac{-3.2+1}{4}\right) = 0.5619$

(c) Conditional probability $P(Z < 1.6/Z > 0.1) =$

Answer: $\frac{\mathbb{P}(0.1<Z<1.6)}{\mathbb{P}(Z>0.1)} = 0.8809$

(d) What is the type (and parameters) of the random variable $Y = 3(2 - X)$?

Answer: $Y = 6 - 3X = 6 - 3(-1 + 4Z) = 9 - 12Z$, so Y is $\mathcal{N}(9, 144)$

2. Let X be an exponential random variable with parameter $\lambda > 0$. Find the **distribution function** and the **density function** for the random variable $Y = 1 - \sqrt{X}$.

Answer: The range is $-\infty < Y \le 1$. Now

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(1 - \sqrt{X} \le y) = \mathbb{P}(X \ge (1-y)^2) = 1 - F_X((1-y)^2) = e^{-\lambda(1-y)^2}$$

and

$$f_Y(y) = F_Y'(y) = 2\lambda(1-y)e^{-\lambda(1-y)^2}$$

both functions for $y < 1$

[Bonus] Find $E(Y)$

Answer:

$$\mathbb{E}(Y) = \int_{-\infty}^{1} 2\lambda y(1-y)e^{-\lambda(1-y)^2}\, dy$$

after changing variable $y = 1 - x$ it becomes

$$\mathbb{E}(Y) = -\int_0^\infty 2\lambda(1-x)xe^{-\lambda x^2}\, dx$$

Integration by parts gives

$$\mathbb{E}(Y) = -(1-x)e^{-\lambda x^2}|_0^\infty - \int_0^\infty e^{-\lambda x^2}\, dx = 1 - \int_0^\infty e^{-\lambda x^2}\, dx$$

The last integral can be computed by reduction to a density of a normal distribution. Denote $\lambda = \frac{1}{2\sigma^2}$, then

$$\int_0^\infty e^{-\lambda x^2}\, dx = \sqrt{2\pi\sigma^2}\,\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^\infty e^{-\frac{x^2}{2\sigma^2}}\, dx$$

and note that

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^\infty e^{-\frac{x^2}{2\sigma^2}}\, dx = \frac{1}{2}$$

Therefore

$$\int_0^\infty e^{-\lambda x^2}\, dx = \frac{\sqrt{2\pi\sigma^2}}{2} = \frac{\sqrt{\pi/\lambda}}{2}$$

and finally

$$\mathbb{E}(Y) = 1 - \frac{\sqrt{\pi/\lambda}}{2}$$

3. Suppose X and Y are two independent random variables such that

$$E(X) = -1, \quad \sigma_X = 2, \qquad E(Y) = -7, \quad \sigma_Y = 3.$$

Let $W = 5 - 3X + Y$. Compute the following:

(a) $E(W) =$

Answer: 0

(b) $\sigma_W =$

Answer: $\mathsf{Var}(W) = (-2)^2 \times 2^2 + 3^2 = 25$, so $\sigma_W = 5$

(c) $E(X^2) =$

Answer: $E(X^2) = \sigma_X^2 + [\mathbb{E}(X)]^2 = 4 + 1 = 5$

(d) $E(Y^2) =$

Answer: $E(Y^2) = \sigma_Y^2 + [\mathbb{E}(Y)]^2 = 9 + 49 = 58$

(e) $E(2X^2 - Y^2 - 3XY) =$

Answer: $E(2X^2 - Y^2 - 3XY) = 2 \times 5 - 58 - 3 \times (-1) \times (-7) = -69$

[Bonus] $E[XY(X - Y)] =$

Answer:

$$\mathbb{E}(X^2Y - XY^2) = \mathbb{E}(X^2Y) - \mathbb{E}(XY^2) = \mathbb{E}(X^2)\mathbb{E}(Y) - \mathbb{E}(X)\mathbb{E}(Y^2) = 23$$

Note: it is not true that $E[XY(X - Y)] = \mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(X - Y)$, because $X - Y$ is not independent of X or Y.

4. Two random variables X and Y have joint density function $f(x, y) = c$ (where $c > 0$ denotes an unknown constant) on the rectangle $\{0 < x < 2,\ 0 < y < 5\}$ (and zero elsewhere).

(a) Find $c =$

Answer: $c = 1/10$

(b) Find $P(X > Y) =$

Answer: $1/5$

(c) Find $P(X^2 + Y^2 \leq 4) =$

Answer: $\pi/10$

(d) Find $P(X = Y) =$

Answer: zero

(e) Find $P(X + Y > 1) =$

Answer: $\frac{19}{20}$

(Bonus) Are X and Y independent? Explain.

yes, because (i) the domain is rectangular and (ii) the joint density function is constant, $f_{X,Y}(x, y) = 1/10$, hence it is equal to the product of the two density functions:

$$f_{X,Y}(x, y) = \frac{1}{10} = \frac{1}{2} \times \frac{1}{5} = f_X(x) \times f_Y(y),$$

which makes X uniform on the interval $(0, 2)$ and Y uniform on the interval $(0, 5)$.

5. Five friends decided to spend a day on the beach. Each one arrive at the beach randomly between 8 AM and 9 AM (with uniform distribution during this hour). Let T_1 be the time when the first one arrives, T_2 be the time of arrival of the second one, etc., so that $T_1 < T_2 < T_3 < T_4 < T_5$ are all five arrival times. (We measure time in hours, i.e., the first arrival at 8:15 AM would be $T_1 = 8.25$, etc)

Note: the arrival time of each person is a uniform random variable on the interval $(8, 9)$, thus its distribution function is

$$F(x) = \frac{x-8}{9-8} = x - 8$$

for $8 < x < 9$

(a) Find the density function of the earliest arrival time, T_1.

Answer: the distribution function is $F_{T_1}(x) = 1 - (1 - F(x))^5 = 1 - (9 - x)^5$, so the density is $f_{T_1}(x) = F'_{T_1}(x) = 5(9 - x)^4$

(b) Find the mean value of T_1, i.e., find $\mathbb{E}(T_1)$

Answer: $\mathbb{E}(T_1) = \int_8^9 5x(9-x)^4\,dx = 8 + \frac{1}{6}$ (to take the integral, it helps to change variable $y = 9 - x$)

(c) Find the density function of the latest arrival time T_5.

Answer: the distribution function is $F_{T_5}(x) = [F(x)]^5 = (x - 8)^5$, so the density is $f_{T_5}(x) = F'_{T_5}(x) = 5(x - 8)^4$

(d) Find the mean value of T_5, i.e., find $\mathbb{E}(T_5)$

Answer: $\mathbb{E}(T_5) = \int_8^9 5x(x-8)^4\,dx = 8 + \frac{5}{6}$ (to take the integral, it helps to change variable $y = x - 8$)

[Bonus] Find all the moments of T_1, i.e., find $E(T_1^k)$ for any $k = 1, 2, \ldots$.

Answer:

$$\mathbb{E}(T_1^k) = \int_8^9 5x^k(9-x)^4\,dx = \int_8^9 5x^k(9^4 - 4\cdot 9^3 x + 6\cdot 9^2 x^2 - 4\cdot 9x^3 + x^4)\,dx$$

the rest is routine calculus, can be omitted

[Bonus] Find the density of the second arrival time T_2.