MA 485-12 (Probability), Chernov — Midterm test #2

Show your work. Each problem is 4 pts. Full credit for 8 problems — Wed, May 7

1. A random variable X takes the following values with the corresponding probabilities:

X	-3	-1	2	3
P	0.1	0.3	0.4	0.2

Sketch the graph of the distribution function $F(x)$. Indicate clearly which endpoints belong to the graph and which do not.

2. A random variable X has the following distribution function:

$$F(x) = \begin{cases} x^3/2 & \text{for } 0 < x \le 1 \\ x/2 & \text{for } 1 < x \le 2 \end{cases}$$

(a) Find the density of X. Answer:

$$f(x) = \begin{cases} 3x^2/2 & \text{for } 0 < x \le 1 \\ 1/2 & \text{for } 1 < x \le 2 \end{cases}$$

(b) $P(X > 1/2) =$ Answer: 15/16

(c) $P(X = 1) =$ Answer: 0

(d) $P(0.8 < X < 1.2) =$ Answer: 0.344

3. By using the attached table of $\Phi(x)$, find the following probabilities. Here X is an $N(0,1)$ and Y is an $N(4,9)$.

(a) $P(X > 2.3) =$ Answer: 0.0107

(b) $P(-1.2 < X < 0.2) =$ Answer: 0.4642

(c) $P(|X| \leq 2.5) =$ Answer: 0.9876

(d) $P(2.5 < Y < 7.3) =$ Answer: 0.5558

4. The lifetime of a VCR is an exponential random variable X with half-life $\bar{t}_{1/2} = 3$ (years).

(a) Find the parameter $\lambda =$ Answer: 0.231

(b) Write down the distribution function $F(x) =$ Answer: $= 1 - e^{-0.231\,x}$ for $x > 0$

(c) Write down the density function $f(x) =$ Answer: $= 0.231\,e^{-0.231\,x}$ for $x > 0$

(d) Find the mode of X: Answer: 0

(e) Compute $P(X > 9) =$ Answer: 0.125

(f) Find conditional probability $P(X > 11/X > 2) =$ Answer: 0.125

5. Let X be an exponential random variable with parameter λ. Find the distribution function $G(x)$ and the density function $g(x)$ of the variable $Y = 1 + 1/X$.

Answer: $G(x) = e^{-\frac{\lambda}{x-1}}$ for $x \geq 1$,

$g(x) = \frac{\lambda}{(x-1)^2}\, e^{-\frac{\lambda}{x-1}}$

6. Two random variables X and Y have joint distribution function $F_{X,Y}(x,y) = xy/4$ for $0 \leq x, y \leq 2$.

(a) Find the joint density function $f_{X,Y}(x,y)$. Answer: $f(x,y) = 1/4$ for $0 \leq x, y \leq 2$

(b) Compute $P(X + Y > 3) =$ Answer: 1/8

(c) Compute the marginal density $f_X(x)$. Answer: $f_X(x) = 1/2$ for $0 < x < 2$ and $f_Y(y) = 1/2$ for $0 < y < 2$

(d) Are the variables X and Y independent? Answer: yes

(e) (Extra credit) Compute $P(X^2 + Y^2 < 1) =$ Answer: $\pi/16$

7. Suppose X and Y are independent random variables uniformly distributed on $(-1, 1)$.

(a) By using the convolution formula find the density of $Z = X + Y$.

Answer: $f_Z(z) = \frac{z+2}{4}$ for $-2 < z < 0$ and $f_Z(z) = \frac{2-z}{4}$ for $0 < z < 2$

(b) Sketch the graph of $f_Z(z)$.

(c) Find the mode of Z. Answer: 0

8. An engine has 10 components. The lifetime (time to failure) of each component is a uniform random variable on the interval $(0, 10)$ (in years). Find the distribution function $F_E(x)$ and the density function $f_E(x)$ of the engine lifetime in the following cases:

(a) The engine fails when one of its components fails.

Answer: $F_E(x) = 1 - (1 - F(x))^{10} = 1 - \left(1 - \frac{x}{10}\right)^{10}$.

$f_E(x) = F'_E(x) = \left(1 - \frac{x}{10}\right)^9$ for $0 < x < 10$

(b) The engine fails when all its components fail.

Answer: $F_E(x) = [F(x)]^{10} = \left(\frac{x}{10}\right)^{10}$.

$f_E(x) = F'_E(x) = \left(\frac{x}{10}\right)^9$ for $0 < x < 10$

(c) [Extra credit] The engine fails when two of its components fail.

Answer: $F_E(x) = 1 - \left(1 - \frac{x}{10}\right)^{10} - 10\left(\frac{x}{10}\right)\left(1 - \frac{x}{10}\right)^9$.

9. In a small town, traffic accidents occur at the rate 1.5 per day. A police officer reports for duty at 8am.

(a) What is the distribution of the waiting time till the first accident on that day? Write its distribution function and density function.

Answer: Exponential with parameter $\lambda = 0.0625$

(b) What is the distribution of the waiting time between the first and the second accidents?

Answer: Same

(c) What is the probability that no accidents occur till 2pm on that day?

Answer: 0.6872