MA 485-12 (Probability), Chernov — Midterm test #2
Show your work. Each problem is 4 pts. — Wed, May 6, 1998

1. A random variable X has the density function $f(x) = \frac{3}{16}x^2$ for $-2 \le x \le 2$. Compute the following:

(a) $EX =$ Answer: $\int_{-2}^{2} x \cdot 3x^2/16\, dx = 0$

(b) $EX^2 =$ Answer: $\int_{-2}^{2} x^2 \cdot 3x^2/16\, dx = 2.4$

(c) $\operatorname{Var} X =$ Answer: 2.4

(d) $\sigma_X =$ Answer: 1.5492

(e) Find the moments of all order $k \ge 1$. Answer: $\int_{-2}^{2} x^k \cdot 3x^2/16\, dx = \frac{3}{16}\frac{2^{k+3}-(-2)^{k+3}}{k+3}$

2. By using the attached table of $\Phi(x)$, find the following probabilities for two normal random variables, $Z = N(0, 1)$ and $X = N(6, 9)$.

(a) $P(Z > 2.14) =$ Answer: 0.0162

(b) $P(-1.66 < Z < 0.52) =$ Answer: 0.65

(c) $P(|Z| \leq 2.73) =$ Answer: 0.9936

(d) $P(4.23 < X < 9.69) =$ Answer: 0.6131

(e) What is the type (and parameters) of the random variable $Y = 7 - 2Z$? Answer: normal, $Y = N(7, 4)$

3. The lifetime of a VCR is an exponential random variable X with half-life $\bar{t}_{1/2} = 6$ (years).

(a) Find the parameter $\lambda =$ Answer: 0.1155

(b) Write down the distribution function $F(x) =$ Answer: $= 1 - e^{-0.1155\,x}$ for $x > 0$

(c) Write down the density function $f(x) =$ Answer: $e^{-0.1155\,x}$ for $x > 0$

(d) Give values for EX and $\operatorname{Var} X$. Answer: 8.66 and 74.96

(e) Compute $P(X > 12) =$ Answer: 0.25

(f) Find the conditional probability $P(X > 15/X > 3) =$ Answer: 0.25

4. Let X be an exponential random variable with parameter $\lambda = 2$. Find the distribution function $F_Y(y)$ and the density function $f_Y(y)$ of the variable $Y = 5 + 2/X$.

For an extra credit, do the same for $Y = 1 - 1/X^2$.

Answer:

$$F_Y(y) = e^{-\frac{4}{y-5}} \qquad \text{for} \quad y > 5$$

$$f_Y(y) = \frac{4}{(y-5)^2}\, e^{-\frac{4}{y-5}} \qquad \text{for} \quad y > 5$$

Extra credit part:

$$F_Y(y) = 1 - e^{-\frac{2}{\sqrt{1-y}}} \qquad \text{for} \quad y < 1$$

$$f_Y(y) = \frac{1}{(1-y)^{3/2}}\, e^{-\frac{2}{\sqrt{1-y}}} \qquad \text{for} \quad y < 1$$

5. Suppose X and Y are two independent random variables such that $EX = 2$, $\mathrm{Var}X = 4$, $EY = -1$ and $\mathrm{Var}Y = 3$. Let $Z = 3X + 4Y - 6$. Compute the following:

(a) $EZ =$ Answer: -4

(b) $\sigma_Z =$ Answer: $\sqrt{84}$

(c) $EX^2 =$ Answer: 8

(d) $EY^2 =$ Answer: 4

(e) $E(3X^2 - 4XY + 2Y^2) =$ Answer: 40

6. A random variable X takes the following values with the corresponding probabilities:

X	-1	0	1	2
P	0.2	0.3	0.4	0.1

Let $Y = X^2$. Compute the following:

(a) $P(X + Y > 1) =$ Answer: 0.5

(b) $EX =$ Answer: 0.4

(c) $EY =$ Answer: 1

(d) $E(XY) =$ Answer: 1

(e) $\text{Cov}(X, Y) =$ Answer: 0.6

7. An engine has 8 components. The lifetime (time to failure) of each component is a uniform random variable on the interval $(0, 5)$ (in years). Find the distribution function $F_E(x)$ and the density function $f_E(x)$ of the engine lifetime in the following cases:

(a) The engine fails when one of its components fails.

Answer: $F_E(x) = 1 - (1 - F(x))^8 = 1 - \left(1 - \frac{x}{5}\right)^8$.

$f_E(x) = F'_E(x) = \frac{8}{5}\left(1 - \frac{x}{5}\right)^7$ for $0 < x < 5$

(b) The engine fails when all its components fail.

Answer: $F_E(x) = [F(x)]^8 = \left(\frac{x}{5}\right)^8$.

$f_E(x) = F'_E(x) = \frac{8}{5}\left(\frac{x}{5}\right)^7$ for $0 < x < 5$

(c) The engine fails when two of its components fail.

Answer: $F_E(x) = 1 - \left(1 - \frac{x}{5}\right)^8 - 8\left(1 - \frac{x}{5}\right)^7\left(\frac{x}{5}\right)$.

$f_E(x) = F'_E(x) = \frac{56}{25}x\left(1 - \frac{x}{5}\right)^6$ for $0 < x < 5$