MA 485-12 (Probability), Chernov | Midterm test #2
Show your work. Each problem is 4 pts. | Wed, Feb 17

1. A random variable $X$ has the density function $f(x) = x/4$ for $1 \le x \le 3$ and 0 otherwise. Compute the following:

(a) $EX =$ Answer: $\int_1^3 x \cdot x/4\, dx = 16/3$

(b) $EX^2 =$ Answer: $\int_1^3 x^2 \cdot x/4\, dx = 5$

(c) $\operatorname{Var} X =$ Answer: $5 - (13/6)^2 \approx 0.3056$

(d) $\sigma_X =$ Answer: $\sqrt{\operatorname{Var} X} \approx 0.553$

(e) Find the moments of all order $k \ge 1$. Answer: $\int_1^3 x^k \cdot x/4\, dx = \frac{3^{k+2}-1}{4(k+2)}$

2. By using the attached table of $\Phi(x)$, find the following probabilities for two normal random variables, $Z = N(0,1)$ and $X = N(6,9)$.

(a) $P(Z > 2.11) =$ Answer: $1 - \Phi(2.11) = .0174$

(b) $P(-1.32 < Z < 0.71) =$ Answer: $\Phi(0.71) - \Phi(-1.32) = 0.6677$

(c) $P(|Z| \leq 2.47) =$ Answer: $\Phi(2.47) - \Phi(-2.47) = 0.9864$

(d) $P(4.02 < X < 9.15) =$ Answer: $F_X(9.15) - F_X(4.02) = \Phi((9.15-6)/3) - \Phi((4.02-6)/3) = \Phi(1.05) - \Phi(-0.66) = 0.5985$

(e) What is the type (and parameters) of the random variable $Y = 12 - 3Z$?
Answer: $\mu = 12$, $\sigma^2 = (-3)^2 = 9$. So, $Y = N(12, 9)$

3. The lifetime of a radioactive atom is an exponential random variable $X$ with half-life $\bar{t}_{1/2} = 12$ (years).

(a) Find the parameter $\lambda =$ Answer: $\lambda = \ln 2/12 = 0.05776$

(b) Write down the distribution function $F(x) =$ Answer: $= 1 - e^{-0.05776\,x}$ for $x > 0$

(c) Write down the density function $f(x) =$ Answer: $= 0.05776\,e^{-0.05776\,x}$ for $x > 0$

(d) Give values for $EX$ and $\operatorname{Var} X$. Answer: $EX = 1/\lambda = 17.31$, $\operatorname{Var} X = 1/\lambda^2 = 299.7$

(e) Compute $P(X > 24) =$ Answer: $1/4$

(f) Find the conditional probability $P(X > 27/X > 3) =$ Answer: $= P(X > 24) = 0.25$ (no memory)

4. Let $X$ be an exponential random variable with parameter $\lambda = 4$. Find the distribution function $F_Y(y)$ and the density function $f_Y(y)$ of the variable $Y = 2 + 6/X$.

Solution: $X > 0$, hence $6/X > 0$, so $Y > 2$.

$F_Y(y) = P(Y \le y) = P(2 + 6/X \le y) = P(6/X \le y - 2)$
$= P(X \ge 6/(y-2)) = 1 - F_X(6/(y-2)) = e^{-24/(y-2)}$

$f_Y(y) = F_Y'(y) = \frac{24}{(y-2)^2} e^{-24/(y-2)^2}$ for $y > 2$

[Bonus] Let $X$ be a uniform random variable on $(0, 1)$. Find the distribution function $F_Y(y)$ and the density function $f_Y(y)$ of the variable $Y = 2 \ln X$.

Solution: $0 < X < 1$, hence $\ln X < 0$, so $Y < 0$.

$F_Y(y) = P(Y \le y) = P(2 \ln X \le y) = P(X \le e^{y/2})$
$= F_X(e^{y/2}) = e^{y/2}$ for $y < 0$

$f_Y(y) = F_Y'(y) = \frac{1}{2} e^{y/2}$ for $y < 0$

5. Suppose $X$ and $Y$ are two independent random variables such that $EX = 4$, $\text{Var}X = 4$, $EY = -6$ and $\text{Var}Y = 3$. Let $Z = 2X - 5Y + 8$. Compute the following:

(a) $EZ =$ Answer: $2 \cdot 4 - 5 \cdot (-6) + 8 = 46$

(b) $\sigma_Z =$ Answer: $\sqrt{91}$

Since $\text{Var}\, Z = 2^2 \cdot \text{Var}\, X + (-5)^2 \cdot \text{Var}\, Y = 4 \cdot 4 + 25 \cdot 3 = 91$

(c) $EX^2 =$ Answer: $\text{Var}\, X + (EX)^2 = 4 + 16 = 20$

(d) $EY^2 =$ Answer: $\text{Var}\, Y + (EY)^2 = 3 + 36 = 39$

(e) $E(3X^2 - 6XY + Y^2) =$ Answer: $3 \cdot 20 - 4 \cdot 4 \cdot (-6) + 2 \cdot 39 = 234$

6. Random variables $X$ and $Y$ are uniform on the interval $(-1, 1)$ and independent. Find the probability that $X^2 + Y^2 < 0.25$.

Answer: $\pi/16$

[Bonus] Find the probability that $|X| + |Y| < 1$.

Answer: $1/2$

7. An engine has 5 components. The lifetime (time to failure) of each component is a uniform random variable on the interval $(0, 10)$ (in years), and their lifetimes are independent. Find the distribution function $F_E(x)$ and the density function $f_E(x)$ of the lifetime of the engine in the following cases:

(a) The engine fails when one of its components fails.

Answer: $F_E(x) = 1 - (1 - F(x))^5 = 1 - \left(1 - \frac{x}{10}\right)^5$.

$f_E(x) = F_E'(x) = \frac{1}{2}\left(1 - \frac{x}{10}\right)^4$ for $0 < x < 10$

(b) The engine fails when all its components fail.

Answer: $F_E(x) = [F(x)]^5 = \left(\frac{x}{10}\right)^5$.

$f_E(x) = F_E'(x) = \frac{1}{2}\left(\frac{x}{10}\right)^4$ for $0 < x < 10$

(c) [Bonus] The engine fails when two of its components fail.

Answer: $F_E(x) = C_{5,3}\left(\frac{x}{10}\right)^3\left(1 - \frac{x}{10}\right)^2 + C_{5,4}\left(\frac{x}{10}\right)^4\left(1 - \frac{x}{10}\right) + C_{5,5}\left(\frac{x}{10}\right)^5$.

$f_E(x) = F_E'(x) = \frac{5!}{2!\,2!}\frac{1}{10}\left(\frac{x}{10}\right)^2\left(1 - \frac{x}{10}\right)^2$ for $0 < x < 10$