MA 485-12, Probability (Dr Chernov) Final Exam Tue, March 7, 2000

Student's name ________________________________

Be sure to show all your work. Each problem is 4 points.

1. A basketball player makes 75% of his free throws on the average. In the following, use de Moivre-Laplace theorem. Apply histogram correction if necessary.

(a) The player attempts free throws repeatedly until he makes 35. What is the probability that at least 49 throws will be necessary?

Answer:

$$P(b(48, 0.75) < 35) \approx P(N(36, 9) < 34.5) =$$

$$= \Phi\left(\frac{34.5 - 36}{3}\right) = \Phi(-0.5) = 0.3085$$

(b) If the player attempts 75 free throws, what is the chance that he will make exactly 57?

Answer:

$$P(b(75, 0.75) = 57) \approx P(56.5 < N(56.25, 14.06) < 57.5) =$$

$$= \Phi\left(\frac{57.5 - 56.25}{3.75}\right) - \Phi\left(\frac{56.5 - 56.25}{3.75}\right) = \Phi(0.333) - \Phi(0.067) = 0.1014$$

2. Let $X_1, \ldots, X_{300}$ be independent random variables uniformly distributed on the interval $(0, 2)$. Let $S = X_1 + \cdots + X_{300}$.

(a) Use the central limit theorem to estimate $P(292 < S < 336)$. Apply histogram correction if necessary.

Answers: $S \approx N(300, 100)$, so

$$P(292 < S < 336) \approx P(292 < N(300, 100) < 336) =$$
$$= \Phi\left(\frac{336 - 300}{10}\right) - \Phi\left(\frac{292 - 300}{10}\right) = \Phi(3.6) - \Phi(-0.08) = 0.7881$$

(b) Use the 3σ-rule to predict the interval in which S takes values. In addition, give the interval of all possible values of S.

Answer: $270 < S < 330$. All possible values: from 0 to 600.

[Bonus] Let $V = X_1 X_2$. Find $\rho_{S,V}$.

3. An investor has a stock that each week goes up \$1 with probability 56% or down \$1 with probability 44%. She bought the stock when it costed \$100 and will sell it when it reaches \$120 or falls to \$90.

(a) What is the probability that she will end up with \$120?

Answer:

$$\frac{1-(44/56)^{10}}{1-(44/56)^{30}} = 0.91$$

(b) Find the mean number of weeks the investor keeps the stock.

Answer:

$$ET = \frac{90 \cdot 0.09 + 120 \cdot 0.91 - 100}{0.56 - 0.44} = 144$$

(c) Assume now that the investor changes the rules: she will only sell the stock when it falls to \$90. Find the probability that she will ever sell the stock.

Answer:

$$\left(\frac{44}{56}\right)^{10} = 0.09$$

4. An investor has a stock that each week goes up \$1 with probability 0.5 or down \$1 with probability 0.5. She bought the stock when it costed \$100 and will sell it when it reaches \$120 or falls to \$90.

(a) What is the probability that she will end up with \$120?

Answer:

$$\frac{100-90}{120-90} = \frac{1}{3}$$

(b) Find the mean number of weeks the investor keeps the stock.

Answer:

$$ET = 10 \cdot 20 = 200$$

(c) Find the probability that the stock will again cost \$100 before the investor sells it. Find the average number of times the price of the stock goes back to \$100 before the investor sells the stock.

Answers:

$$P = 1 - \frac{30}{2 \cdot 20 \cdot 10} = 0.925$$

$$G = \frac{0.925}{0.075} = 12.33$$

5. Assume that accidents on a 1000 miles long highway occur at a rate of one accident per 40 miles (on average). Peter is driving on this highway.

(a) Peter covers the first 20 miles and notices no accidents. What is the probability that in the next 20 miles, Peter will notice at least one accident? Give the formula for the probability that in the next 20 miles Peter will notice exactly k accidents.

Answers: Assume the unit of length be one mile. Then $\lambda = 0.025$. We have $N_{20} =$ poisson(0.5).

$$P(N_{20} \geq 1) = 1 - e^{-0.5} = 0.3937$$

$$P(N_{20} = k) = \frac{(0.5)^k}{k!} e^{-0.5}$$

(b) Let X be the total number of accidents on the entire 1000 miles of the highway. What is the type of the random variable X? What is its parameter? Use normal approximation to compute $P(23 < X < 35)$. Apply histogram correction if necessary.

Answers: $X =$ poisson(25) $\approx N(25, 25)$.

$$P(23 < X < 35) \approx \Phi\left(\frac{34.5 - 25}{5}\right) - \Phi\left(\frac{23.5 - 25}{5}\right) = \Phi(1.9) - \Phi(-0.3) = 0.5892$$

6. Assume that accidents on a 1000 miles long highway occur at a rate of one accident per 40 miles (on average).

(a) What is the distribution of the intervals between accidents? Write down a formula for the distribution function, give its mean value and variance. State clearly which unit of length you are using.

Answers: $F(x) = 1 - e^{-0.025\,x}$, $EX = 40$, $\operatorname{Var} X = 1600$.

(b) Let V be the number of accidents on the first 600 miles of the highway and W the number of accidents on the last 800 miles of the highway (see the sketch below). Note that these two intervals of the highway overlap. Compute the covariance $\operatorname{Cov}(V, W)$ and the correlation coefficient $\rho_{V,W}$.

Answers: Note that the two parts of the highway overlap by 400 miles. Now, $\operatorname{Cov}(V, W) = 10$ and

$$\rho_{V,W} = \frac{10}{\sqrt{15} \cdot \sqrt{20}} \approx 0.578$$

7. Suppose the weight of a certain brand of bolt has mean of 1 gram and a standard deviation of 0.05 gram.

(a) Use the central limit theorem to estimate the probability that 900 of these bolts weigh more than 903 grams. Use histogram correction if necessary.

Answer: $S_{900} = N(900, (1.5)^2)$. Hence

$$P(S_{900} > 903) = 1 - \Phi\left(\frac{903 - 900}{1.5}\right) = 1 - \Phi(2) = 0.0228$$

(b) Let W be the weight of 900 of these bolts. Use Chebyshev's inequality to estimate $P(|W - 900| \geq 15)$.

Answer:

$$P(|W - 900| \geq 15) \leq \frac{(1.5)^2}{225} = 0.01$$

8. Suppose that guilty people fail a lie detector test with probability 0.95, and innocent people pass a lie detector test with probability 0.96. The police arrest five people, of which one commited a crime (and the police do not know which one). One arrested person is selected at random for a lie detector test.

(a) If the lie detector says that the person is guilty, what is the probability that he really is?

Answer:

$$\frac{0.2 \times 0.95}{0.2 \times 0.95 + 0.8 \times 0.04} = 0.856$$

(b) If the lie detector says that the person is innocent, what is the probability that he really is?

Answer:

$$\frac{0.8 \times 0.96}{0.8 \times 0.96 + 0.2 \times 0.05} = 0.987$$

[Bonus] Suppose all the five arrested people take the lie detector test, and X of them fail (marked guilty by the detector). Find EX and $\operatorname{Var} X$.

9. Suppose X and Y are independent and uniform on $(0,1)$. Let $V = X^3$ and $W = Y - X$.

(a) Find EV and $\operatorname{Var} V$. [Recall the formula for EX^k where $X = U(0,1)$.]

Answers: $EV = 1/4$ and $\operatorname{Var} V = 1/7 - 1/16$.

(b) Find EW and $\operatorname{Var} W$.

Answers: $EW = 0$ and $\operatorname{Var} W = 1/12 + 1/12 = 1/6$.

(c) Find $\operatorname{Cov}(V,W)$ and $\rho_{V,W}$. How do you interpret the value of $\rho_{V,W}$, including its sign?

Answer:

$$\operatorname{Cov}(V,W) = E(V \cdot W) = EX^3Y - EX^4 = \frac{1}{4} \cdot \frac{1}{2} - \frac{1}{5} = -\frac{3}{40}$$

10. A continuous random variable X has distribution function $F(x) = \sqrt{x/4}$ for $0 < x < 4$. In the problems below, do not forget to find the range of the random variable Y.

(a) Find the distribution and density function of $Y = 1 + 2/\sqrt{X}$.

Answers: $Y > 2$,

$$F_Y(y) = 1 - \frac{1}{y-1}$$

$$f_Y(y) = \frac{1}{(1-y)^2}$$

(b) Find the distribution and density function of $Y = \ln X$.

Answers: $Y < \ln 4$,

$$F_Y(y) = \frac{1}{2}\,e^{y/2}$$

$$f_Y(y) = \frac{1}{4}\,e^{y/2}$$

[Bonus] Find the distribution and density function of $Y = (X-2)^2$.