MA 485-12, Probability (Dr Chernov) Final Exam Thu, March 8, 2001

Student's name ______________________________

Be sure to show all your work. Each problem is 4 points.

1. Let $X = N(2,5)$ and $Y = N(3,4)$ be two independent normal random variables

(a) Find $P(X > Y)$.

Solution: $X - Y = N(-1,9)$, so

$$P(N(-1,9) > 0) = 1 - \Phi\left(\frac{0-(-1)}{3}\right) = 0.3707$$

(b) Find $P(X < 7 - 2Y)$.

Solution: $X + 2Y = N(8,21)$, so

$$P(N(8,21) < 7) = \Phi\left(\frac{7-8}{\sqrt{21}}\right) = 0.4129$$

[Bonus] Find $P(X^2 + Y^2 > 3 - 2XY)$

Hint: $P((X+Y)^2 > 3) = P(|X+Y| > \sqrt{3}) = 1 - P(-\sqrt{3} < X + Y < \sqrt{3})$

2. A certain brand of batteries have average lifetime 150 hours and standard deviation 10 hours.

(a) What can be said about the percentage of batteries with lifetime at most 90 hours?

Solution: $P(X \leq 90) \leq P(|X - 150| \geq 60) \leq 10^2/60^2 = 1/36$

(b) What can be said about the percentage of batteries with lifetime at least 200 hours?

Solution: $P(X \geq 200) \leq P(|X - 150| \geq 50) \leq 10^2/50^2 = 1/25$

3. A random variable X has moment-generating function

$$M_X(t) = \frac{1}{(t-1)^2}$$

(a) Find $E(X)$ (b) Find $E(X^2)$ (c) Find $\mathrm{Var}(X)$ (d) Find $E(X^3)$

Solution:

$$E(X) = M_X'(0) = 2$$
$$E(X^2) = M_X''(0) = 6$$
$$\mathrm{Var}(X) = 6 - 2^2 = 2$$
$$E(X^3) = M_X'''(0) = 24$$

[Bonus] Recall that if X_1, X_2 are independent random variables, then $M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t)$. Then try to figure out how to represent X as a sum of two independent random variables. What are they?

Solution:

$$M_X(t) = \frac{1}{1-t} \cdot \frac{1}{1-t}$$

so $X = X_1 + X_2$, where X_1, X_2 are exponential(1).

4. Let $X_1, \ldots, X_{50}$ be independent random variables exponentially distributed with parameter $\lambda = 5$. Let $S = X_1 + \cdots + X_{50}$. Use the central limit theorem to estimate $P(8 < S < 11)$ and $P(S > 3)$. Apply histogram correction if necessary.

Solution: $E(X_1) = 1/5$, $\mathrm{Var}(X_1) = 1/25$, $E(S) = 10$, $\mathrm{Var}(S) = 2$.

$$P(8 < S < 11) = \Phi\left(\frac{11-10}{\sqrt{2}}\right) - \Phi\left(\frac{8-10}{\sqrt{2}}\right) = 0.6818$$

$$P(S > 3) = 1 - \Phi\left(\frac{3-10}{\sqrt{2}}\right) = 1$$

5. An investor has a stock that each week goes up \$10 with probability 50% or down \$10 with probability 50%. She bought the stock when it costed \$600 and will sell it when it reaches \$1000 or falls to \$300.

(a) What is the probability that she will end up with \$1000?

Solution: Note: unit=10.

$$P_b = \frac{60 - 30}{100 - 30} = \frac{3}{7}$$

(b) Find the mean number of weeks the investor keeps the stock.

Solution: $E(T) = (60 - 30)(100 - 60) = 1200$

(c) Find the probability that the stock will again cost \$600 before the investor sells it. Find the average number of times the price of the stock goes back to \$600 before the investor sells the stock.

Solution:

$$P(60, 60) = 1 - \frac{100 - 30}{2(100 - 60)(60 - 30)} = 0.971$$

$$G(60, 60) = \frac{0.971}{1 - 0.971} \approx 33$$

6. Suppose X and Y are independent random variables uniformly distributed on the interval $(0, 1)$. Let $V = X^2 + 1$ and $W = Y - X^2$.

(a) Find $E(V)$ and $\mathrm{Var}(V)$.

Answers: $E(V) = \frac{4}{3}$ and $\mathrm{Var}(V) = \frac{4}{45}$.

(b) Find $E(W)$ and $\mathrm{Var}(W)$.

Answers: $E(W) = \frac{1}{6}$ and $\mathrm{Var}(V) = \frac{31}{180}$.

(c) Find $\mathrm{Cov}(V, W)$ and $\rho_{V,W}$. How do you interpret the value of $\rho_{V,W}$, including its sign?

Answers: $\mathrm{Cov}(V, W) = -\frac{4}{45}$ and $\rho_{V,W} = -0.718$.

[Bonus] Find a general formula for $E[(X + Y)^n]$ for all $n \geq 1$

7. Suppose you buy cereal of a certain brand, and 60% of boxes with cereal contain a coupon. When you collect 20 such coupons, you can order a free backpack by mail. In the following, use De Moivre-Laplace theorem.

(a) What is the probability that you will need to buy at least 40 boxes of cereal to collect 20 coupons?

Solution: 39 boxes are not enough, so

$$P(b(39, 0.6) < 20) = P(N(23.4, 9.36) < 19.5) = \Phi\left(\frac{19.5 - 23.4}{\sqrt{9.36}}\right) = 0.1020$$

(b) What is the probability that if you buy 100 boxes of cereal, then you get exactly 60 coupons?

Solution:

$$P(b(100, 0.6) = 60) = P(59.5 < N(60, 24) < 60.5)$$
$$= \Phi\left(\frac{60.5 - 60}{\sqrt{24}}\right) - \Phi\left(\frac{59.5 - 60}{\sqrt{24}}\right) = 0.0796$$

8. Assume that accidents on a highway occur at a rate of one accident per 25 miles (on average). Paul is driving on this highway.

(a) Paul covers the first 60 miles and notices 2 accidents. What is the probability that in the next 75 miles Peter will notice at least two more accidents? Give the formula for the probability that in the next 160 miles Paul will notice exactly k accidents.

Solution: rate=1/25. $N_{75} =$ poisson(3), so

$$P(N_{75} \geq 2) = 1 - p(0) - p(1) = 1 - e^{-3} - 3e^{-3} = 0.8008$$

(b) What is the distribution of the intervals between accidents? Write down formulas for the distribution function and moment-generating function. Give the mean value and variance for the intervals between accidents.

Solution: $W =$ exponential(0.04),

$$F(x) = 1 - e^{-0.04 \cdot x} \qquad M_X(t) = \frac{0.04}{0.04 - t}$$

$$E(W) = 25 \qquad \mathrm{Var}(W) = 625$$

9. David has \$1000 in his bank account. Each week, he either withdraws \$100 with probability 0.55 or deposits \$100 with probability 0.45.

(a) What is the probability that David's bank account will ever grow to \$2000?

Solution: $p = 0.45$, $q = 0.55$. Note: unit=100.

$$P_b = \frac{1 - (0.55/0.45)^{10}}{1 - (0.55/0.45)^{20}} = 0.1185$$

(b) In how many weeks, on average, will the account dry up (i.e., drop to 0)?

Solution. Note: it is a one-sided random walk.

$$E(T) = \frac{10 - 0}{0.55 - 0.45} = 100$$

10. A continuous random variable X has distribution function

$$F(x) = x^2 \qquad \text{for} \ \ 0 < x < 1$$

(a) Find the distribution and density function of $Y = -\ln X$.

Solution: $Y > 0$,

$$F_Y(y) = P(\ln X \geq -y) = P(X \geq e^{-y}) = 1 - e^{-2y}$$

$$f_Y(y) = 2e^{-2y}$$

[Bonus] Find the distribution and density function of $V = X(1 - X)$.