MA 485-1E, Probability (Dr Chernov) Final Exam Wed, Dec 12, 2001

Student's name ______________________________

Be sure to show all your work. Each problem is 4 points. Full credit will be given for 9 problems (36 points). You are welcome to do more for extra credit.

1. Let X be a nonnegative random variable with $E(X) = 80$ and $\sigma_X = 10$. What can you say about $P(X > 100)$? Use Markov's inequality and Chebyshev's inequality to answer this question. Which one gives a better estimate?

Solution: By Markov inequality: $P(X > 100) \leq 0.8$

By Chebyshev's inequality: $P(X > 100) \leq P(|X - 80| > 20) \leq 100/20^2 = 0.25$

The second estimate is better.

2. Suppose X and Y are independent random variables uniformly distributed on the interval $(0, 1)$. Let $V = X - 10Y$. Find $\mathrm{Cov}(Y, V)$ and $\rho_{Y,V}$. How do you interpret the value of $\rho_{Y,V}$, including its sign?

[Bonus] Find a general formula for $E[(X + Y)^n]$ for all $n \geq 1$

Answer:

$$\mathrm{Cov}(Y, X - 10Y) = \mathrm{Cov}(Y, X) - 10\,\mathrm{Vax}(X) = 0 - 10 \cdot (1/12) = -5/6$$

$$\mathrm{Var}(Y) = 1/12$$

$$\mathrm{Var}(V) = \mathrm{Var}(C) + 100\,\mathrm{Var}(Y) = (1/12) + 100 \cdot (1/12) = 101/12$$

$$\rho_{Y,V} = \frac{-5/6}{\sqrt{1/12} \cdot \sqrt{101/12}} = -0.995$$

Y and V are strongly negatively correlated.

3. A certain brand of batteries have average lifetime 50 hours and standard deviation 10 hours. Find the (approximate) chance that the total (combined) lifetime of 20 randomly selected batteries will exceed 1010 hours.

We use the cetral limit theorem. We have $\mu = 50$, $\sigma = 10$, and $n = 20$. Hence

$$P(S_{20} > 1010) \approx 1 - \Phi\left(\frac{1010 - 1000}{\sqrt{2000}}\right) = 1 - \Phi(0.22) = 0.4129$$

4. Donna puts 100 one dollar bills in a jar. Each day, she rolls a die, and if it shows 1, 2 or 3, she takes two bills from the jar, and if the die shows 4, 5 or 6, Donna puts two more singles into the jar. She decides to stop the game if the jar gets empty or if it reaches \$400.

(a) Find the probability that the jar gets empty.

Answer: the unit step is \$2, hence $a = 0, b = 200, x = 50$. Now

$$P_0 = \frac{200 - 50}{200 - 0} = 0.75$$

(b) Find the mean number of days Donna's game lasts.

Answer: $E(T) = 50 \cdot 150 = 7500$.

(c) Determine the probability that the initial situation (i.e., the jar contains exactly \$100) ever repeats, and find the average number of repetitions.

Answer:

$$P(50, 50) = 1 - \frac{200 - 0}{2(200 - 50)(50 - 0)} = 0.9867$$

$$G(50, 50) = \frac{0.9867}{1 - 0.9867} = 74$$

5. Assume that accidents on a highway occur at a rate of one accident per 10 miles (on the average). Roger is driving on that highway.

(a) Roger covers the first 60 miles and notices 8 accidents. What is the probability that in the next 30 miles Roger will notice at least three more accidents?

Answer: let the unit length be one mile. Then $\lambda = 1/10$, $b - a = 30$ and $\lambda(b-a) = 3$. So,

$$P(N \geq 3) = 1 - P(N = 0) - P(N = 1) - P(N = 2)$$

$$= 1 - e^{-3} - 3e^{-3} - \frac{9}{2}e^{-3} = 0.5768$$

(b) What is the distribution of the intervals between accidents (in miles)? Write down formulas for the distribution function and the moment-generating function.

Answer: W is exponential with parameter $\lambda = 0.1$, so

$$F_W(x) = 1 - e^{-0.1\,x} \qquad x \geq 0$$

$$M_W(t) = \frac{0.1}{0.1 - t}$$

6. Sandy puts 50 one dollar bills in a jar. Each day, she rolls a die, and if it shows one or two, she takes five bills from the jar, and if the die shows 3, 4, 5 or 6, Sandy puts five new singles into the jar. She decides to stop the game if the jar gets empty or if it reaches \$150.

(a) Find the probability that the jar gets empty.

Answer: the unit step is \$5, hence $a = 0, b = 30, x = 10$. So,

$$P_0 = \frac{(1/2)^{10} - (1/2)^{30}}{1 - (1/2)^{30}} = 0.000977$$

(b) Find the mean number of days Sandy's game will last.

Answer:

$$E(T) = \frac{0 + 30(1 - 0.00097) - 10}{2/3 - 1/3} = 59.91$$

7. Suppose that customers arrive at a small convenience store at a rate of one customer per three minutes. Let X be the total number of customers coming to the store between 8 AM and noon on a certain day. Find the probability $P(70 < X < 100)$. (You may use normal approximation to a Poisson random variable.)

Answer: $X =$poisson(80). By the central limit theorem, $X \approx Y = N(80, 80)$. Hence,

$$P(70 < X < 100) \approx P(70.5 < Y < 99.5) = \Phi\left(\frac{99.5 - 80}{\sqrt{80}}\right) - \Phi\left(\frac{70.5 - 80}{\sqrt{80}}\right) = 0.8408$$

[Bonus] Let Y be the number of customers coming to that store between 11 AM and 1 PM on the same day. Find $\mathrm{Cov}(X, Y)$.

Answer: $\mathrm{Cov}(X, Y) = 10$.

8. Suppose that 80% of the patients with a certain desease can be cured with a certain drug. A doctor decides to prescribe this drug to his patients until it fails on 10 of them, and then switch to another drug. In the following, use De Moivre-Laplace theorem.

(a) What is the probability that at least 35 of the doctor's patients will be successfully cured before the doctor switches to another drug?

Solution: the condition "at least 35 will be successfully cured before 10 fail" means that at least 35+10=45 patients will be treated with this drug before it fails on 10 of them. Hence, 44 treatments will not be enough to have 10 failures. Let X be the number of failures after 44 treatments. Then $X = \text{binomial}(44, 0.2)$ and

$$P(X \leq 9) \approx \Phi\left(\frac{9.5 - 8.8}{\sqrt{7.04}}\right) = 0.6026$$

(b) What is the probability that, out of 100 patients, exactly 75 can be cured with this drug?

Solution: let X be the number of patients successfully cured out of 100. Then $X = \text{binomial}(100, 0.8)$ and

$$P(X = 75) \approx \Phi\left(\frac{75.5 - 80}{\sqrt{16}}\right) - \Phi\left(\frac{74.5 - 80}{\sqrt{16}}\right) = 0.0454$$

9. Suppose X and Y are two independent random variables such that $E(X) = 2$, $E(Y) = -1$, $\sigma_X = 1$, and $\sigma_Y = 2$. Let $V = 3 - 2X + Y$. Compute the following:

(a) $EV =$ Answer: -2

(b) $\text{Var}(V) =$ Answer: 8

(b) $\sigma_V =$ Answer: $2\sqrt{2}$

(c) $EX^2 =$ Answer: 5

(d) $EY^2 =$ Answer: 5

(e) $E[(X+Y)(X-2Y)] =$ Answer: -3

(Bonus) $\text{Cov}(V, X - Y) =$ Answer: -6

10. Let $X = N(5,5)$ and $V = N(-1,1)$ be two independent normal random variables and $Y = 3 + 2V$.

(a) Find the type (and parameters) of the variable Y.

Answer: $Y = N(1,4)$.

(b) Find $P(X > Y)$.

Answer: $X - Y = N(4,9)$, so

$$P(X > Y) = P(X - Y > 0) = 1 - \Phi\left(\frac{0-4}{3}\right) = 0.9082$$

(b) Find $P(Y < 2 + 2X)$.

Answer: $Y - 2X = N(-9,24)$, so

$$P(Y < 2 + 2X) = P(Y - 2X < 2) = \Phi\left(\frac{2-(-9)}{\sqrt{24}}\right) = 0.9878$$

(Bonus) Find $P(X^2 + 4Y^2 > 9 - 4XY)$.

11. A random variable X has moment-generating function is given by the following formula:

$$M_X(t) = \frac{1}{3} + \frac{1}{6}e^{-6t} + ce^t(2 + e^{-t})$$

(a) Find the value of c.

Solution: we have

$$M_X(t) = \frac{1}{3} + c + \frac{1}{6}e^{-6t} + 2ce^t$$

Since the coefficients must add up to one, we need

$$\frac{1}{3} + c + \frac{1}{6} + 2c = 1$$

hence $c = 1/6$ and

$$M_X(t) = \frac{1}{2} + \frac{1}{6}e^{-6t} + \frac{1}{3}e^t$$

(b) Find $E(X)$.

Answer:

$$E(X) = M_X'(0) = -1 + 1/3 = -2/3 = -0.667$$

(c) Find $\mathrm{Var}(X)$.

Answer:

$$E(X) = M_X'(0) = 6 + 1/3 = 6.333$$

so

$$\mathrm{Var}(X) = 6.333 - (-0.667)^2 = 5.888$$