MA 485-1E, Probability (Dr Chernov) Final Exam Wed, Dec 11, 2002

Student's name ____________________

Be sure to show all your work. Full credit is given for 100 points.

1. (12 pts) The average price of gasoline is currently \$1.38 per gallon. Assume that each week it goes up 2 cents with probability 53% or down 2 cents with probability 47%.

(a) Find the probability that the average price per gallon goes down to \$1 before it reaches \$2.

Solution: Our unit is \$0.02. Now $a = 50, b = 100, x = 69$, and

$$P_{50} = \frac{(0.47/0.53)^{19} - (0.47/0.53)^{50}}{1 - (0.47/0.53)^{50}} \approx 0.0998$$

(b) Find the average number of weeks the price will stay between \$1 and \$2.

Solution:

$$E(T) = \frac{50 \times 0.0998 + 100 \times 0.9002 - 69}{0.53 - 0.47} \approx 433.5$$

(c) Find the probability that the price ever drops below \$1 (without any upper restriction).

Solution:

$$P_{50} = (0.47/0.53)^{19} \approx 0.102$$

2. (14 pts) Let $X$ be a Poisson random variable with mean 50.

(a) Use Markov's inequality to obtain an upper bound on $p = P(X \geq 75)$.

Solution: $P(X \geq 75) \leq 50/75 = 2/3$.

(b) Use Chebyshev's inequality to obtain an upper bound on $p$.

Solution:

$$P(X \geq 75) \leq P(|X - 50| \geq 25) \leq 50/25^2 = 0.08$$

(c) Compute $p$ approximately by using the central limit theorem.

Solution: $X \approx Y = N(50, 50)$, and

$$P(X \geq 75) \approx P(Y \geq 74.5) = 1 - \Phi\left(\frac{74.5 - 50}{\sqrt{50}}\right) \approx 0.0003$$

(note histogram correction!)

3. (10 pts) Suppose $X_i$ for $i = 1, 2, 3, 4$ are independent random variables with mean $\mu$ and standard deviation $\sigma$. Let $V = X_1 - X_2 + X_3$ and $W = X_2 - X_3 + X_4$.

(a) Find $\text{Cov}(V, W)$.

Solution:

$$\text{Cov}(V, W) = -\text{Var}(X_2) - \text{Var}(X_3) = -2\sigma^2$$

(b) Find $\rho_{V,W}$. How do you interpret the value of $\rho_{V,W}$, including its sign?

Solution:

$$\rho_{V,W} = \frac{-2\sigma^2}{\sqrt{3\sigma^2}\sqrt{3\sigma^2}} = -\frac{2}{3}$$

Strongly negatively correlated (typical fluctuations occur in opposite directions).

4. (10 pts) An insurance company has 40,000 automobile policyholders. The expected yearly claim per person is \$250 with a standard deviation of \$900.

(a) Compute the probability that the total yearly claim exceeds \$10.5 mln.

Solution: $E(S) = 40,000 \times 250 = 10,000,000$, $\text{Var}(S) = 40,000 \times 900^2 = 3.24 \times 10^{10}$. Hence

$$P(S \geq 10,500,000) \approx 1 - \Phi\left(\frac{10,500,000 - 10,000,000}{\sqrt{3.24 \times 10^{10}}}\right) \approx 0.0027$$

(b) Use the $3\sigma$-rule to predict the total yearly claim with probability 99.7%.

Solution: $\sigma_S = 180,000$, hence

$$9,460,000 < S < 10,540,000$$

5. (12 pts) Jim and Jake play poker. After every game the loser pays the winner a dollar. Initially, Jim has \$15 and Jake has \$18. They are equally good at poker and each wins any given game with probability 50%. They have plenty of time and will play until one of them runs out of money.

(a) Find the probability that Jim wins in the end.

Solution: $x = 15, a = 0, b = 15 + 18 = 33$, hence

$$P_{33} = \frac{15 - 0}{33 - 0} = 0.454$$

(b) Find the mean number of games they will play.

Answer: $E(T) = 15 \times 18 = 270$.

(c) Find the probability that Jim's fortune will come back to exactly \$15 before it is all over.

Solution:

$$P(15, 15) = 1 - \frac{33}{2 \times 18 \times 15} = 0.939$$

(d) How many times, on average, Jim's fortune will come back to \$15?

Solution:

$$G(15, 15) = \frac{0.939}{1 - 0.939} \approx 15.4$$

6. (10 pts) A die is continually rolled until the total sum of all rolls exceeds 100. What is the probability that at least 35 rolls are necessary? (Use the central limit theorem. Assume that the variance of each roll is 2.92, see 11.4 (c) in the classnotes.)

Solution: First, we have $E(X_i) = 3.5$ and $\text{Var}(X_i) = 2.92$. Now, the event "at least 35 rolls are necessary" is equivalent to "34 rolls are not enough", that is $S_{34} \le 100$. By the central limit theorem

$$P(S_{34} \le 100) \approx P(Y \le 100.5)$$

where $Y = N(34 \times 3.5, 34 \times 2.92) = N(119, 99.28)$, and we used histogram correction. Finally,

$$P(Y \le 100.5) = \Phi\left(\frac{100.5 - 119}{\sqrt{99.28}}\right) \approx 0.0314$$

7. (12 pts) Suppose that, on a given day, fires in a large city occur randomly at any location and make a two-dimensional Poisson process with a rate of $\lambda = 0.5$ fires per square mile (see sections 17.11–17.14 in the classnotes).

(a) A fire station must respond to any fire within the radius of 2 miles around it. What is the probability that on that day it will respond to more than one fire? (That is, what is the probability that at least two fires occur within two miles of the fire station?)

Solution: Area is $A = 0.5(\pi \times 2^2) = 2\pi$, so $N =$poisson$(2\pi)$. Now

$$P(N \geq 2) = 1 - P(N = 0) - P(N = 1) = 1 - e^{-2\pi} - 2\pi e^{-2\pi} \approx 0.986$$

(b) Let $X$ be the distance from the fire station to the nearest fire. Find the distribution function of $X$. Find the density function of $X$.

Solution:

$$F_X(x) = P(X \leq x) = 1 - P(N = 0) = 1 - e^{-0.5\pi x^2}$$

and

$$f_X(x) = F_X'(x) = \pi x e^{-0.5\pi x^2}$$

8. (10 pts) The moment-generating function of a random variable $X$ is given by the following formula:

$$M_X(t) = \frac{1}{2} + c(e^t + e^{-t})^2$$

(a) Find the value of $c$.

Answer: $c = 1/8$.

(b) Find $E(X)$.

Answer: $E(X) = 0$.

(c) Find $\text{Var}(X)$.

Answer: $\text{Var}(X) = 1$.

[Bonus] Find the third and the fourth moments of $X$.

Answer: $E(X^3) = 0$ and $E(X^4) = 4$.

9. (10 pts) A system has 10 components. The lifetime (time to failure) of each component is an exponential random variable with half-life $t_{1/2} = 2$, and their lifetimes are independent. Find the **distribution function** $F_S(x)$ and the **density function** $f_S(x)$ of the lifetime of the system in the following cases:

(a) It is a parallel ("robust") system, i.e. it functions as long as at least one component works.

Answer: note that $\lambda = \ln 2/2$. Now

$$F(x) = \left(1 - e^{-\lambda x}\right)^{10}$$

and

$$f(x) = 10\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{9}$$

(b) It is a series ("fragile") system, i.e. it functions as long as all components work.

Answer:

$$F(x) = 1 - e^{-10\lambda x}$$

and

$$f(x) = 10\lambda e^{-10\lambda x}$$

[Bonus] It is a "3-out-of-10 system", i.e. it works as long at least 3 components work.

Answer:

$$F(x) = \sum_{i=0}^{2} C_{10,i} e^{-i\lambda x}\left(1 - e^{-\lambda x}\right)^{10-i}$$

and

$$f(x) = 10\, C_{9,2} \lambda e^{-3\lambda x}\left(1 - e^{-\lambda x}\right)^{7}$$

10. (10 pts) Let $X = N(-2, 3)$ and $Y = N(1, 1)$ be two independent normal random variables and $V = 3Y - 3$.

(a) Find the type and parameters of the variable $V$.

Answer: $V = N(0, 9)$.

(b) Find $P(X < V)$.

Solution: $P(X < V) = P(X - V < 0)$, and $X - V = N(-2, 12)$. Hence

$$P(X - V < 0) = \Phi\left(\frac{0 + 2}{\sqrt{12}}\right) \approx 0.7190$$

[Bonus] Find $P(X^2 + V^2 > 9 - 2XV)$.

Hint:

$$P((X + V)^2 > 9) = P(|X + V| > 3) = P(X + V > 3) + P(X + V < -3)$$