MA 485-1E, Probability (Dr Chernov) Final Exam Wed, Dec 10, 2003

Student's name ______________________________

Be sure to show all your work. Full credit is given for 100 points.

1. (10 pts) Assume that accidents on a 600 miles long highway occur at a rate of one accident per 20 miles (on average).

(a) A motorist takes the highway and notices 4 accidents within the first 50 miles. What is the probability that the motorist will notice at least two accidents in the next 50 miles?

Answer: the number of accidents on a 50 miles stretch is $N$=poisson($\lambda$) with $\lambda = 50/20 = 2.5$. Hence

$$P(N > 1) = 1 - P(N = 0) - P(N = 1) = 1 - e^{-2.5} - 2.5\, e^{-2.5} = 0.7127$$

(b) Let $X$ be the total number of accidents on the entire 600 miles of the highway. What is the type of the random variable $X$? What is its parameter? Use normal approximation to compute $P(28 < X < 37)$. Apply histogram correction if necessary.

Answer: $X$=poisson(30) $\approx Y = N(30, 30)$, hence

$$P(28 < X < 37) \approx P(28.5 < Y < 36.5) = \Phi(1.187) - [1 - \Phi(0.27)] = 0.4894$$

2. (10 pts) Let $X = U(0, 12)$. Estimate the probability $P(X > C)$ assuming that $6 < C < 12$:

(a) by Markov inequality;

Answer: $P(X > C) < 6/C$.

(b) by Chebyshev inequality.

Answer: $P(X > C) < 12/(C - 6)^2$.

(c) Which inequality gives a better estimate for $C = 11$?

Answer: $P(X > C) < 6/11$ by Markov and $P(X > C) < 12/25$ by Chebyshev, hence the latter is better.

[Bonus] Find all the values of $C$ for which Markov inequality gives a better estimate.

3. (12pts) An investor has a stock that each week goes 50 cents up with probability 49% or 50 cents down with probability 51%. She bought the stock when it costed \$15 and will sell it when it reaches \$20 or falls to \$1.

(a) What is the probability that she will end up with \$20?

Answer: here $a = 2$, $b = 40$, $x = 30$, $p = 0.49$, $q = 0.51$, hence

$$P_b = \frac{1 - (0.51/0.49)^{28}}{1 - (0.51/0.49)^{38}} = 0.578$$

(b) Find the mean number of weeks the investor keeps the stock.

Answer:

$$\frac{2 \times 0.422 + 40 \times 0.578 - 30}{0.49 - 0.51} = 301.8$$

(c) Assume there is no lower bound on the stock price (i.e. it is allowed to take negative values) and find the probability that the price of the stock will ever reach \$20.

Answer: $(0.49/0.51)^{10} = 0.67$.

4. (10 pts) (a) A fair die is continually rolled until a six comes up 11 times. Use De Moivre-Laplace theorem to compute the probability that at least 73 rolls will be necessary.

Answer: $X = b(72, 1/6) \approx Y = N(12, 10)$, hence

$$P(X < 11) \approx P(Y < 10.5) = \Phi(-0.47) = 0.3192$$

(b) Compute the probability that in the first 144 rolls a six comes up exactly 23 times.

Answer: $X = b(144, 1/6) \approx Y = N(24, 20)$, hence

$$P(X = 23) \approx P(22.5 < Y < 23.5) = \Phi(-0.11) - \Phi(-0.34) = 0.0893$$

5. (10 pts) Max opens a bank account and puts \$10,000 in it. Each week, he either withdraws \$200 with probability 0.5 or deposits \$200 with probability 0.5. Max will close his account if the balance drops to zero.

(a) What is the probability that Max's bank account will ever grow to \$30,000?

Answer: here $a = 0$, $b = 150$, $x = 50$, $p = q = 0.5$, hence

$$P_b = \frac{50 - 0}{150 - 0} = \frac{1}{3}$$

(b) Suppose Max does not close the account as long as the balance is positive, then how many weeks, on average, will the account remain open?

Answer: $E(T) = \infty$.

(c) Find the probability that the balance will again be \$10,000 before it reaches \$30,000.

Answer:

$$P(x, x) = 1 - \frac{150 - 0}{2(150 - 50)(50 - 0)} = 0.985$$

(d) How many times, on average, will the balance return to its initial value of \$10,000 before it reaches \$30,000?

Answer:

$$G(x, x) = \frac{0.985}{0.015} = 65.67$$

6. (12 pts) Let $X_1, \dots, X_{450}$ be independent random variables uniformly distributed on the interval $(0, 1)$. Let $S = X_1^2 + \cdots + X_{450}^2$. (Note the squares!)

(a) Use the central limit theorem to estimate $P(143 < S < 156)$. Apply histogram correction if necessary.

Answer: here $\mu = E(X^2) = 1/3$ and $\sigma^2 = E(X^4) - [E(X^2)]^2 = 4/45$. Hence $S \approx Y = N(150, 40)$.

$$P(143 < S < 156) \approx P(143 < Y < 156) = \Phi(0.95) - \Phi(-1.11) = 0.6932$$

(b) Use the $3\sigma$-rule to predict the interval in which $S$ takes values. In addition, give the interval of all possible values of $S$.

Answer: $(150 - 3\sqrt{40}, 150 + 3\sqrt{40}) = (131, 169)$.

7. (12 pts) A multicomponent system has 20 components. The lifetime (time to failure) of each component is a random variable with distribution function

$$F(x) = 1 - x^{-2}, \quad x > 1$$

and their lifetimes are independent. Find the **distribution function** $F_S(x)$ and the **density function** $f_S(x)$ of the lifetime (time to failure) of the entire system in the following cases:

(a) The system is functioning only if all its components work.

Answer:

$$F(x) = 1 - x^{-40}, \quad f(x) = 40x^{-41}$$

(b) The system is functioning as long as at least one component works.

Answer:

$$F(x) = (1 - x^{-2})^{20}, \quad f(x) = 40(1 - x^{-2})^{19} x^{-3}$$

(c) The system is functioning as long as at least three components work.

Answer:

$$F(x) = \sum_{i=0}^{2} C_{20,i}\, x^{-2i} (1 - x^{-2})^{20-i}$$

and

$$f(x) = 40\, C_{19,2}\, x^{-7} (1 - x^{-2})^{17}$$

8. (12 pts) Let $X = N(7, 8)$ and $Y = N(3, 8)$ be two independent normal random variables.

(a) Find $P(X < Y + 3)$.

Answer: $X - Y = N(4, 16)$, hence

$$P(X - Y < 3) = \Phi(-0.25) = 0.4013$$

(b) Let $V = 2X + Y$ and $W = X - Y$. Find $\mathrm{Cov}(V, W)$ and $\rho_{V,W}$.

Answer: $\mathrm{Cov}(V, W) = 8$, $\rho_{V,W} = 8/\sqrt{640} = 0.3162$

9. (12 pts) The moment-generating function of a random variable $X$ is given by the following formula:

$$M_X(t) = c + \sum_{n=1}^{\infty} \frac{e^{nt}}{10^n}$$

(a) Show that $M_X(t) = c + \frac{e^t}{10-e^t}$ whenever $e^t < 10$.

Answer: this is a geometric series with the common ratio $e^t/10$:

$$M_X(t) = c + (e^t/10) \sum_{n=0}^{\infty} (e^t/10)^n = c + \frac{e^t/10}{1 - e^t/10} = c + \frac{e^t}{10 - e^t}$$

(b) Find the value of $c$. Is it larger than 0.88?

Answer: $M_X(0) = c + 1/9 = 1$, hence $c = 8/9$.

(c) Find all the values of the random variable $X$ and their probabilities. Is it related to any random variables that we studied?

Answer: values: $0, 1, 2, \ldots$ (all nonnegative integers). Probabilities:

$$P(X = 0) = \frac{8}{9}, \quad P(X = n) = \frac{1}{10^n} \text{ for } n \geq 1.$$

[Bonus] Derive a formula for the $k$-th moment of the random variable, i.e. for $E(X^k)$.