MA 485-4A, Probability (Dr Chernov) Final Exam Thu, Dec 9, 2004

Student's name ______________________________

Be sure to show all your work.

Each problem is 14 points (112 total!). Full credit is given for 100 points!

1. Two random variables X and Y have joint density function $f(x, y) = c$ on the domain $\{x > 0,\ y > 0,\ x + y < 2\}$ (and zero elsewhere).

(a) Find $c =$ Answer: 1/2

(b) Find $P(X > Y) =$ Answer: 1/2

(c) Find $P(X^2 + Y^2 < 1) =$ Answer: $\pi/8$

(d) Find $P(X = Y) =$ Answer: 0

(e) Find $P(X - Y > 2) =$ Answer: 0

(f) For $0 < x < 2$, find $P(X < x) =$ Answer: $1 - (2 - x)^2/2$

(g) For $0 < y < 2$, find $P(Y < y) =$ Answer: $1 - (2 - y)^2/2$

(Bonus) Find the density functions of X and Y Answer: $2 - x$ and $2 - y$

(Bonus) Are X and Y independent? Answer: no

2. An investor has a stock that each week goes 25 cents up with probability 50% or 25 cents down with probability 50%. She bought the stock when it costed \$45 and will sell it when it reaches \$50 or falls to \$35.
(a) What is the probability that she will end up with \$50?

Answer: 2/3

(b) Find the mean number of weeks the investor keeps the stock.

Answer: 800

(c) Find the probability that the price of the stock will again be \$45 before the investor sells it.

Answer: 0.9625

(d) How many times, on average, will the price of the stock return to its initial value of \$45 before the investor sells it?

Answer: 25.667

3. Let $X = N(1,3)$ and $Y = N(-2,4)$ be two independent normal random variables.
(a) Find $P(2X < Y + 3)$.

Answer: $1 - \Phi(0.25) = 0.4013$.

(b) Use Chebyshev's inequality to estimate $P(Y > 8)$.

Answer: $P(Y > 8) = P(Y + 2 > 10) \leq P(|Y + 2| > 10) < 0.04$.

(Bonus) Estimate, as best as you can, the actual probability $P(Y > 8)$ (Note: answers like 1 or 0 are not good enough).

$P(Y > 8) = 1 - \Phi(5) \approx 7 \times 10^{-7}$.

4. Suppose you buy cereal of a certain brand, and 70% of boxes of cereal contain a coupon. When you collect 30 such coupons, you can order a free baseball cap by mail.

(a) What is the probability that you will need to buy at least 41 boxes of cereal to collect 30 coupons?

Answer: $\Phi(0.52) = 0.6985$

(b) What is the probability that if you buy 200 boxes of that cereal, then you get exactly 140 coupons?

Answer: $\Phi(0.08) - \Phi(-0.08) = 0.0638$.

(Bonus) What is the probability that if you buy $10n$ boxes of cereal, then you get exactly $7n$ coupons? Investigate this probability as $n \to \infty$.

Answer:

$$2\,\Phi\left(\frac{0.5}{\sqrt{2.1\,n}}\right) - 1 \approx 2\,\Phi'(0)\,\frac{0.5}{\sqrt{2.1\,n}} = \frac{1}{\sqrt{4.2\,\pi n}}$$

5. Let $Z = N(0,1)$ be a standard normal random variable. It is known that

$$E(Z^4) = 3, \qquad E(Z^6) = 15, \qquad E(Z^8) = 105$$

(a) Use the moment generating function of Z to verify that $E(Z^4) = 3$.
(b) Assuming the above values for the moments of Z, compute $\mathrm{Var}(Z^2)$ and $\mathrm{Var}(Z^4)$.

Answer: $\mathrm{Var}(Z^2) = 2$ and $\mathrm{Var}(Z^4) = 96$

(c) Assuming the above values for the moments of Z, compute the covariance $\mathrm{Cov}(Z^2, Z^4)$ and the correlation coefficient ρ_{Z^2,Z^4}.

Answer: $\mathrm{Cov}(Z^2, Z^4) = 12$, $\rho_{Z^2,Z^4} = 12/\sqrt{192} = 0.866$

(Bonus) Use the moment generating function of Z to verify that $E(Z^6) = 15$.

6. Let $Z_1, Z_2, \ldots, Z_n$ be independent standard normal random variables, and let

$$X = Z_1^2 + Z_2^2 + \cdots + Z_n^2$$

(this is called the χ^2 random variable with n degrees of freedom).
(a) Use the previous problem to compute $E(X)$ and $\text{Var}(X)$.

Answer: $E(X) = n$ and $\text{Var}(X) = 2n$

(b) Use the central limit theorem to find a normal approximation to X.

Answer: $X \approx N(n, 2n)$.

(c) Approximate the probability $P(X < n + \sqrt{n})$.

Answer: $\Phi(0.71) = 0.7611$

7. Suppose X is a continuous random variable with density function

$$f(x) = \begin{cases} 3x^2 & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the distribution and density function for $Y = 1 - X^2$.
(Bonus) Find the mean value $E(Y)$.

Answer:

$$F(y) = 1 - \sqrt{1-y}$$
$$f(y) = \frac{3}{2}\sqrt{1-y}$$
$$E(Y) = \frac{2}{5}$$

8. Suppose that mushrooms in a forest make a two-dimensional Poisson process with a rate of $\lambda = 2$ mushrooms per square yard (see sections 17.11–17.14 in the class-notes).
(a) If you find a mushroom, what is the probability that there is another mushroom within 18 inches from it?

Answer: $P(N \geq 1) = 1 - e^{-\pi/2} = 0.792$

(b) Let X denote the number of mushrooms in a lot of size 5×10 yards. Describe the random variable X (type, parameters). Approximate the probability $P(X > 98)$.

Answer:
$X =$ poisson(100)
$P(X > 98) = 1 - \Phi(-0.15) = 0.5596.$

(Bonus) Find the average distance from a randomly selected point in the forest to the nearest mushroom.