MA 485-1C (Probability), Dr. Chernov — Final Exam

7 problems, 15 points for each. Full credit is 100 points. — Wed, Dec 14, 2011

1. (15 pts) Assume that customers arrive at a small store at a rate of one customer per 5 minutes (on average). John works at the store from 8 AM till 1 PM and Bob works there from 12 noon till 6 PM. Let X be the number of customers coming to the store during John's shift and Y be the number of customers coming to the store during Bob's shift.

(a) Find the following:

$$\mathbb{E}(X) = 60 \qquad\qquad \mathsf{Var}(X) = 60$$

(b) Use normal approximation to compute $\mathbb{P}(54 \le X < 63)$. Apply the histogram correction if necessary:

Solution: X is poisson(60), approximated by normal $Y = \mathcal{N}(60, 60)$, so

$$\mathbb{P}(54 \le X < 63) \approx \mathbb{P}(53.5 < Y < 62.5) = \Phi\Big(\frac{62.5 - 60}{\sqrt{60}}\Big) - \Phi\Big(\frac{53.5 - 60}{\sqrt{60}}\Big)$$
$$= \Phi(0.32) - \Phi(-0.84) = 0.6255 - (1 - 0.7995) = 0.4250.$$

(c) Find $\mathsf{Cov}(X, Y) = 12$

(d) Find the correlation coefficient $\rho_{X,Y} = \frac{12}{\sqrt{66}\cdot\sqrt{72}} = 0.183$

Are X and Y independent? **Yes** or **No**? Is the dependence strong or weak?

(e) Find $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 60 + 72 = 132$

(f) Find $\mathsf{Var}(X + Y) = \mathsf{Var}(X) + \mathsf{Var}(Y) + 2\mathsf{Cov}(X, Y) = 60 + 72 + 2 \cdot 12 = 156$

2. (15 pts) Suppose that 85% of patients with a certain disease can be cured with a certain drug. In the following, use De Moivre-Laplace theorem:

(a) What is the probability that, out of 100 patients, exactly 84 will be cured with this drug?

Solution: X is binomial(100,0.85), approximated by normal $Y = \mathcal{N}(85, 12.75)$, so

$$\mathbb{P}(84 \leq X \leq 84) \approx \mathbb{P}(83.5 < Y < 84.5) = \Phi\Big(\frac{84.5 - 85}{\sqrt{12.75}}\Big) - \Phi\Big(\frac{83.5 - 85}{\sqrt{12.75}}\Big)$$
$$= \Phi(-0.14) - \Phi(-0.42) = 0.1071.$$

(b) Now suppose a doctor decides to prescribe this drug to his patients until it fails on 5 of them, and then switch to another drug. What is the probability that at least 36 of the doctor's patients will be successfully cured before the doctor switches to another drug?

Solution: the condition "at least 36 will be successfully cured before 5 fail" means that at least 36+5=41 patients will be treated with this drug before it fails on 5 of them. Hence, 40 treatments will not be enough to have 5 failures. Let X be the number of failures after 40 treatments. Then X is binomial$(40, 0.15)$ and it is approximated by normal $Y = \mathcal{N}(6, 5.1)$, so

$$\mathbb{P}(X \leq 4) \approx \mathbb{P}(Y < 4.5) = \Phi\Big(\frac{4.5 - 6}{\sqrt{5.1}}\Big) = \Phi(-0.66) = 0.2546.$$

3. (15 pts) Let $X_1, \ldots, X_4$ be independent values of a uniform random variable $X = U(0,2)$, i.e., X is distributed over the interval $(0,2)$. Let $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq X_{(4)}$ be the corresponding order statistics.

(a) Find the density function of $X_{(1)}$:

Distribution function $F(x) = 1 - (1 - \frac{x}{2})^4$.
Density function $f(x) = F'(x) = 2(1 - \frac{x}{2})^3$.

(b) Find the density function of $X_{(4)}$:

Distribution function $F(x) = (\frac{x}{2})^4$.
Density function $f(x) = F'(x) = 2(\frac{x}{2})^3$.

(c) Find the density function of $X_{(2)}$:

Distribution function $F(x) = 6(\frac{x}{2})^2(1 - \frac{x}{2})^2 + 4(\frac{x}{2})^3(1 - \frac{x}{2}) + (\frac{x}{2})^4$.
Density function $f(x) = F'(x) = 3x(1 - \frac{x}{2})^2$.

[Bonus] Find the mean value of $X_{(1)}$:

$\mathbb{E}(X_{(1)}) = \int_0^2 2x(1 - \frac{x}{2})^3 \, dx = \frac{2}{5}$.

4. (15 pts) An investor has a stock that each week goes 50 cents up with probability 50% or 50 cents down with probability 50%. She bought the stock when it costed \$25 and will sell it when it reaches \$55 or falls to \$5.

(a) What is the probability that she will sell it at \$5?

Answer: 0.6.

(b) Find the mean number of weeks the investor keeps the stock.

Answer: 2400.

(c) Find the probability that the price of the stock will again be \$25 before the investor sells it.

Answer: 0.979.

(d) How many times, on average, will the price of the stock return to its initial value of \$25 before the investor sells it?

Answer: 47.

5. (15 pts) Let $X = \mathcal{N}(6,9)$ and $Y = \mathcal{N}(2,4)$ be two independent normal random variables.

(a) Find $\mathbb{P}(X > 3 + 2Y)$

Solution: $X - 2Y$ is normal $\mathcal{N}(2,25)$, so

$$\mathbb{P}(X > 3 + 2Y) = \mathbb{P}(X - 2Y > 3) = 1 - \Phi\Big(\frac{3-2}{5}\Big) = 1 - \Phi(0.2) = 0.4207.$$

(b) Use Chebyshev's inequality to estimate $\mathbb{P}(X > 30)$:

Solution:

$$\mathbb{P}(X > 30) = \mathbb{P}(X - 6 > 30 - 6) \le \mathbb{P}(|X - 6| > 24) \le \frac{\mathsf{Var}(X)}{24^2} = \frac{9}{24^2} = \frac{1}{64}.$$

[Bonus] Estimate, as best as you can, the actual probability $\mathbb{P}(X > 30)$ (Note: answers like 1 or 0 are not good enough; use the respective formula).

Solution:

$$\mathbb{P}(X > 30) = 1 - \Phi(8) \approx \frac{1}{8\sqrt{2\pi}} e^{-\frac{8^2}{2}} \approx 6.3 \times 10^{-16}.$$

[Bonus] Find $\mathbb{P}(X^2 + 4Y^2 > 9 + 4XY)$

Solution:

$$\mathbb{P}(X^2 + 4Y^2 > 9 + 4XY) = \mathbb{P}((X - 2Y)^2 > 9)$$
$$= \mathbb{P}(X - 2Y > 3) + \mathbb{P}(X - 2Y < -3) = 1 - \Phi(0.2) + \Phi(1) = 0.5794.$$

6. (15 pts) The moment-generating function of a random variable X is given by the following formula:

$$M_X(t) = \frac{1}{5} + c(e^{2t} + e^{-t})^2$$

(a) Find the value of c:

Answer: $c = 1/5$.

(b) Find all possible values of X and the corresponding probabilities:

Answer: X takes values $-2, 0, 1, 4$ with probabilities $0.2, 0.2, 0.4, 0.2$, respectively.

(c) Find $\mathbb{E}(X) = \frac{4}{5}$

[Bonus] Find $\mathsf{Var}(X) = \frac{94}{25}$

7. (15 pts) Let $X_1, X_2, \ldots, X_{112}$ be independent random variables uniformly distributed over the interval $(0,1)$.

(a) Recall the formula for $\mathbb{E}(X_1^k)$ for each $k \geq 1$:

$\mathbb{E}(X_1^k) = \frac{1}{k+1}$.

(b) Find $\mathsf{Var}(X_1^3) = \frac{9}{112}$

Now let $S = X_1^3 + \cdots + X_{112}^3$ (note the cubes!).

(c) Use the central limit theorem to estimate $P(26 < S < 30)$. Apply histogram correction if necessary:

Solution: S is approximately normal $Y = \mathcal{N}(28, 9)$, so

$$\begin{aligned}\mathbb{P}(26 < S < 30) \approx \mathbb{P}(26 < Y < 30) &= \Phi\left(\frac{26-28}{3}\right) - \Phi\left(\frac{30-28}{3}\right) \\ &= \Phi(0.67) - \Phi(-0.67) = 0.4792.\end{aligned}$$