MA 485-12 (Probability), Chernov | Final exam
. | Wed, May 29

Student name____

1. Two players play chess making a \$1 bet in each game. The player A (skillful) has initually three dollars and probability 0.4 of winning any given game. The player B (rich) has initially \$101 and probability 0.2 of winning any given game. The probability of a draw in any game is 0.4, and in the case of a draw the players simply continue playing.

Find the probability that A eventually loses his \$3 and goes broke. [First, estimate it intuitively, what is your guess?] Find the expected number of games. [Note: $2^{104} = 4 \cdot (2^{10})^{10} \approx 4 \cdot (10^3)^{10} = 4 \cdot 10^{30}$.]

Also solve this problem assuming that the player B is infinitely rich.

For an extra credit, find the probability that the initial situation (A has \$3 and B has \$101) will repeat and find the average number of repetitions.

2. The random variables X_1 and X_2 are independent, and both are normal with parameters $\mu_{X_1} = 5$, $\sigma^2_{X_1} = 2$ and $\mu_{X_2} = -3$, $\sigma^2_{X_2} = 2$. What are the density functions of the variables $Y = X_1 + 2X_2$ and $Z = 2X_1 - X_2$? Find the covariance of Y and Z, verify that Y and Z are independent.

For an extra credit, find the distribution function function of $(Y - \mu_Y)^2 + (Z - \mu_Z)^2$. (Recall a similar example done in class.)

3. Major earthquakes in the USA occur randomly, with the average rate of 2.5 earthquakes every 10 years. Suppose that in 1995 two earthquakes occured.

(a) What was the probability of that (i.e., that two earthquakes occur in a given year)?

(b) What is the expected (mean) waiting time till the next earthquake after the last (second) earthquake in 1995? What is the distribution function of that waiting time?

(c) If there is no earthquake in the first six months of 1996, then what are the chances that this year will be free of earthquakes?

4. A fair die is rolled 180 times. Let N be the number of rolls in which a six appears. Use normal approximation and name the theorems you employ to estimate

(a) $P(N = 30)$

(b) $P(N > 32)$

(c) Predict the range of the random variable N based on the 'three sigma' rule.

5. A random variable X has the following distribution function:

$$F(x) = \begin{cases} \frac{1}{3}\sqrt{x} & \text{for } \ 0 < x < 1 \\ \frac{1}{3}x & \text{for } \ 1 \leq x < 3 \end{cases}$$

(a) Determine whether X is continuous or not.

(b) Find the density of X

$f(x) =$

(c) $P(0.25 < X < 1.2) =$

(d) $P(X = 1) =$

(e) $P(X < 1.2 | X > 0.25) =$

(extra credit) Find the mode, median and both quartiles of the variable X.

6. The random variables $X_1, \ldots, X_{75}$ are independent and uniformly distributed on the interval $(-1, 1)$. Find EX_i and $\operatorname{Var} X_i$ for all $1 \leq i \leq 75$. Estimate

$$P(|X_1 + X_2 + \cdots + X_{75}| \geq 10)$$

Use normal approximation and name the theorem you employ.

7. Mushrooms grow randomly in a forest, with the average density of 0.5 mushrooms per square meter. If you enter the forest and find a mushroom, then what is the probability that within a meter of that spot there is another mushroom?

For an extra credit, find the mean distance to the nearest mushroom from any given point in the forest. (Recall a similar example done in class.)

8. In a lottery, 1% of lottery entries win. Suppose you buy 25 lottery entries. Then what are your chances to win? What are the chances that two entries out of your 25 win? [Name the method you use.]

For an extra credit, how many entries do you need to buy so that at least one will win with probability 99%.

9. A computer motherboard contains 6 chips. The motherboard works as long as all 6 chips are functioning. The lifetime of any given chip is an exponential random variable with the half-life of 8.32 years, and the lifetimes of the chips are independent. [Note: $8.32 \approx 12 \ln 2$.] Let T be the trouble-free life time of the computer motherboard (until some chip goes down and has to be replaced).

Find the distribution function $F_T(x)$. Find the density $f_T(x)$. Find ET and σ_T.

For an extra credit, suppose that one chip goes down and a new (identical) one is installed in its place. Then what are the chances that the next broken chip will be one of the original chips rather than the one just installed?

10. The random variables X and Y are uniformly distributed on the interval $(-1, 1)$ and independent.

(a) Use the convolution formula to find the density function of the random variable $Z = X + Y$.

(b) Find the joint density function $f_{X,Y}(x, y)$. Then find the probability $P(X^2+Y^2 \leq 1)$. [For an extra credit, based on the result of your calculations, can you suggest how to evaluate the number π by a computer that has a random number generator producing numbers between -1 and 1?]

11. The random variable X has an exponential distribution with parameter λ. Find the distribution function and the density function for each of the following variables. (Be sure to find the range for the variable Y.)

(a) $Y = 1 + \frac{1}{X}$.

(b) $Y = X^2 + 1$.

(extra credit) $Y = \ln X$.