MA 485-12, Probability (Dr Chernov) Final Exam Wed, June 3, 1998

Student's name ____________________

Be sure to show all your work. Every problem is worth 4 points.

1. Two players, A and B, play a game betting \$1 each time. The player A is more skillful and wins each game with probability 0.6 (and loses with probability 0.4). The player A starts with \$4, and the player B with \$80. They play until one of them hits zero. Find the probability P_A that A hits zero (loses his four dollars). Find the mean number of games. Next, assume that the player B is infinitely rich, and redo the problem.

Answer: $a = 0$, $b = 84$, $x = 4$, $p = 0.6$, $q = 0.4$, so

$$P_0 = \frac{(0.4/0.6)^4 - (0.4/0.6)^{84}}{1 - (0.4/0.6)^{84}} \approx 0.1975$$

$$P_{84} = 1 - P_0 = 0.8025$$

$$ET = \frac{0 \cdot 0.1975 + 84 \cdot 0.8025 - 4}{0.6 - 0.4} \approx 317$$

If $b = \infty$, then

$$P_0 = (0.4/0.6)^4 \approx 0.1975$$

$$ET = \infty$$

For an extra credit, assume that both A and B are equally skillful and win with probability 0.5. Let again A have \$4 and B have \$80, initially. Determine the probability that the initial situation (i.e. that A has \$4) ever repeats, and find the average number of repetitions.

Answers:

$$P_0 = \frac{84 - 4}{84 - 0}$$

$$ET = 80 \cdot 4 = 320$$

$$P(4,4) = 1 - \frac{84 - 0}{2 \cdot 80 \cdot 4} = 0.869$$

$$G(4,4) = \frac{0.869}{1 - 0.869} \approx 6.6$$

2. For two independent random variables, X and Y, it is known that $EX = -3$, $\sigma_X = 3$, and $EY = 4$, $\sigma_Y = 2$. Let $U = 5 - 2Y + X$ and $V = 2X + 2 - Y$.

(a) Compute EU and EV

Answers:

$$EU = 5 - 2 \cdot 4 + (-3)$$

$$EV = 2 \cdot (-3) + 2 - 4$$

(b) Compute $\operatorname{Var} U$ and $\operatorname{Var} V$.

Answers: $\operatorname{Var} U = 25$, $\operatorname{Var} V = 40$

(c) Compute $\operatorname{Cov}(U, V)$ and the correlation coefficient $\rho_{U,V}$.

Answers: $\operatorname{Cov}(U, V) = 26$,

$$\rho_{U,V} = \frac{26}{\sqrt{25 \cdot 40}}$$

3. Assume that fires in a city occur at a rate 14 per week.

(a) If there was no fire yesterday, what is the probability the there will be no fire today?

Answers: Unit = 1 day. $\lambda = 14/7 = 2$. Then $P(\text{no fire}) = P(N = 0) = e^{-2}$

(b) What is the distribution of the waiting time till the next fire? Write down formulas for the distribution function, give its mean value and variance. State clearly what time units you are using.

Answers: W =exponential(2), $F_W(x) = 1 - e^{-2x}$ for $x > 0$. Then $EW = 1/2$, $\operatorname{Var} W = 1/4$

(c) Let X be the number of fires in a month (30 days). What is the type of the random variable X and its parameter? By using normal approximation, compute $P(X > 55)$. (Do not forget histogram correction.)

Answers: X =poisson(60) $\approx N(60, 60)$, so

$$P(X > 55) = 1 - \Phi\left(\frac{55.5 - 60}{\sqrt{60}}\right)$$

4. A student knows 25% of answers in an oral test. The professor keeps asking questions until the student answers 10 of them correctly.

(a) What is the probability that more than 50 questions will be necessary? (Use the central limit theorem.)

Answer:

$$P(b(49, 0.25) < 10) \approx \Phi\left(\frac{9.5 - 49 \cdot 0.25}{\sqrt{49 \cdot 0.25 \cdot 0.75}}\right)$$

(b) What is the probability that if the professor asks exactly 50 questions, the student answers exactly 10 right?

Answer:

$$P(b(50, 0.25) = 10) \approx P(9.5 < N(12.5, 9.375) < 10.5) =$$
$$= \Phi\left(\frac{10.5 - 12.5}{\sqrt{9.375}}\right) - \Phi\left(\frac{9.5 - 12.5}{\sqrt{9.375}}\right)$$

5. Mushrooms grow in a forest randomly, with density 0.5 per square yard.

(a) If you find a mushroom, what is the chance that at least one more will be within one yard from it? What is the chance that there is exactly one mushroom within the distance one yard from the point you stay?

(b) Let X be the distance to the nearest mushroom from the points you stay. Find the distribution function and the density function of X. (Do not just copy the formula from the class notes, provide calculation.) For an extra credit, compute the mean value of X.

6. Suppose that guilty people are detected by a lie detector with probability 0.95, and innocent people appear to be guilty with probability 0.01. The police arrest 10 suspects, only one of which is guilty (and you do not know which one). A suspect is selected at random for a lie detector test.

(a) If the lie detector says that the suspect is guilty, what is the probability that he really is?

Answer:

$$\frac{0.1 \times 0.95}{0.1 \times 0.95 + 0.9 \times 0.01}$$

(b) If the lie detector says that the suspect is innocent, what is the probability that he really is?

Answer:

$$\frac{0.9 \times 0.99}{0.9 \times 0.99 + 0.1 \times 0.05}$$

7. Let $X_1, \ldots, X_{400}$ be i.i.d. random variables, uniformly distributed on $(0,1)$. Let $\bar{X} = (X_1 + \cdots + X_{400})/400$. Use the central limit theorem to estimate $P(0.48 < \bar{X} < 0.54)$.

Answers: $\bar{X} \approx N(0.5, (12 \cdot 400)^{-1})$, so

$$P(0.48 < \bar{X} < 0.54) \approx \Phi\left(\frac{0.54 - 0.5}{\sqrt{1/4800}}\right) - \Phi\left(\frac{0.48 - 0.5}{\sqrt{1/4800}}\right)$$

8. A random variable X has distribution function $F(x) = x^2$ for $0 < x < 1$. In the problems below, do not forget to indicate the range of the random variables Y and Z.

(a) Find the distribution and density function of $Y = 5 - X^2$.

Answers: $4 < Y < 5$,

$$F_Y(y) = y - 4$$

$$f_Y(y) = 1$$

(b) Find the distribution and density function of $Z = 1 + \ln X$.

Answers: $-\infty < Z < 1$,

$$F_Z(z) = e^{2z-2}$$

$$f_Z(z) = 2e^{2z-2}$$

9. Let X and Y be independent random variables with densities $f_X(x) = 1$ for $0 < x < 1$ and $f_Y(x) = 2x$ for $0 < x < 1$. Use the convolution formula to find the density $f_Z(y)$ for $Z = X + Y$. Graph this density.