MA 485-12, Probability (Dr Chernov) Final Exam Mon, March 15, 1999

Student's name ______________________

Be sure to show all your work. Every problem is worth 4 points.

1. John is buying lottery tickets knowing that each ticket wins with probability 0.2.

(a) If John buys 100 tickets, what is the probability that at least 20 tickets win?

Answer:

$$P(b(100, 0.2) \geq 20) = 1 - P(b(100, 0.2) \leq 19) \approx 1 - P(N(20, 16) < 19.5) =$$

$$= 1 - \Phi\left(\frac{19.5 - 20}{4}\right) = 1 - \Phi(-0.125) \approx 0.5517$$

(b) If John buys 100 tickets, what is the probability that exactly 20 tickets win?

Answer:

$$P(b(100, 0.2) = 20) \approx P(19.5 < N(20, 16) < 21.5) =$$

$$= \Phi\left(\frac{20.5 - 20}{4}\right) - \Phi\left(\frac{19.5 - 20}{4}\right) = \Phi(0.125) - \Phi(-0.125) = 0.1034$$

(c) If John buys tickets repeatedly until 10 tickets win, what is the probability that 49 tickets will not be enough?

Answer:

$$P(b(49, 0.2) < 10) \approx \Phi\left(\frac{9.5 - 49 \cdot 0.2}{\sqrt{49 \cdot 0.2 \cdot 0.8}}\right)$$

$$= \Phi\left(-0.107\right) = 0.4562$$

2. Suppose that guilty people are detected by a lie detector with probability 0.97, and innocent people appear to be guilty with probability 0.02. The police arrest 20 suspects, and three of them are guilty (were involved in the crime). A suspect is selected at random from those arrested for a lie detector test.

(a) If the lie detector says that the suspect is guilty, what is the probability that he really is?

Answer:

$$\frac{0.15 \times 0.97}{0.15 \times 0.97 + 0.85 \times 0.02} = 0.895$$

(b) If the lie detector says that the suspect is innocent, what is the probability that he really is?

Answer:

$$\frac{0.85 \times 0.98}{0.85 \times 0.98 + 0.15 \times 0.03} = 0.9946$$

(c) [Bonus] Suppose all the 20 arrested people take the lie detector test, and X of them fail (marked guilty by the detector). Find EX and $\operatorname{Var} X$.

Answer:

$$X = b(3, 0.97) + b(17, 0.02) \quad \text{(independent)}$$

$$EX = 3 \cdot 0.97 + 17 \cdot 0.02 = 3.25$$

$$\operatorname{Var} X = 3 \cdot 0.97 \cdot 0.03 + 17 \cdot 0.02 \cdot 0.98 = 0.4205$$

3. Let $X_1, \dots, X_{100}$ be i.i.d. random variables, uniformly distributed on $(0,1)$. Let $\bar{X} = (X_1 + \cdots + X_{100})/100$.

(a) Use the central limit theorem to estimate $P(0.47 < \bar{X} < 0.52)$.

Answers: $\bar{X} \approx N(0.5, (12 \cdot 100)^{-1})$, so

$$P(0.47 < \bar{X} < 0.52) \approx \Phi\left(\frac{0.52 - 0.5}{\sqrt{1/1200}}\right) - \Phi\left(\frac{0.47 - 0.5}{\sqrt{1/1200}}\right)$$
$$= \Phi(0.69) - \Phi(-1.04) = 0.6057$$

(b) Use Chebyshev's inequality to estimate $P(|\bar{X} - 0.5| \geq 0.1)$.

Answer:

$$P(|\bar{X} - 0.5| > 0.1) \leq \frac{\operatorname{Var} \bar{X}}{(0.1)^2} = \frac{1}{12}$$

(c) Use the 3σ-rule to predict the interval in which $\bar{X}$ takes values. What are all the possible values for $\bar{X}$?

Answer:

$$0.5 - 3/\sqrt{1200} < \bar{X} < 0.5 + 3/\sqrt{1200}$$

or

$$0.4134 < \bar{X} < 0.5866$$

4. Kim puts 50 one dollar bills in a jar. Each day, she rolls a die, and if it shows one or two, she takes \$1 from the jar, and if the die shows 3, 4, 5 or 6, Kim puts another dollar into the jar. She decides to stop the game if the jar gets empty or if it reaches \$1,000. Find the probability that the jar gets empty. Find the mean number of days the game lasts.

Answer: $a = 0$, $b = 1000$, $x = 50$, $p = 2/3$, $q = 1/3$, so

$$P_0 = \frac{(1/2)^{50} - (1/2)^{1000}}{1 - (1/2)^{1000}} \approx 10^{-15}$$

$$ET = \frac{0 \cdot 10^{-15} + 1000 \cdot (1 - 10^{-15}) - 50}{2/3 - 1/3} = 2850$$

Assume now that Kim changes the rules: she never stops the game unless the jar gets empty. Find the probability that the game will ever stop.

Answer: $\approx 10^{-15}$

5. Meagan puts 50 one dollar bills in a jar. Each day, she rolls a die, and if it shows 1, 2 or 3, she takes \$1 from the jar, and if the die shows 4, 5 or 6, Meagan puts another dollar into the jar. She decides to stop the game if the jar gets empty or if it reaches \$1,000. Find the probability that the jar gets empty. Find the mean number of days the game lasts.

Answer:

$$\frac{950-0}{1000-0} = 0.95$$

$$ET = 50 \cdot 950 = 47500$$

Determine the probability that the initial situation (i.e., the jar contains exactly \$50) ever repeats, and find the average number of repetitions.

Answers:

$$P = 1 - \frac{1000}{2 \cdot 50 \cdot 950} = 0.9895$$

$$G = \frac{0.9895}{0.0105} = 95.2$$

6. Assume that accidents on a 100 miles long highway occur at a rate of 0.02 per mile. Bob is driving on this highway.

(a) Bob drives the first 50 miles and notices one accident. What is the probability that in the next 50 miles, Bob will notice more than one accident?

Answers: $\lambda = 0.02$. We have $N_{50} =$poisson(1).

$$P(N_{50} \geq 2) = 1 - p(0) - p(1) = 1 - e^{-1} - 1 \cdot e^{-1} = 0.264$$

(b) Let X_s be the number of accidents on a stretch of the highway of length s (miles). What is the distribution of the random variable X_s? Find $P(X_s = 0)$.

Answer: X_s =poisson$(0.02 \cdot s)$ and

$$P(X_s = 0) = e^{-0.02\,s}$$

(c) Let X be the number of accidents in the first 60 miles of the highway and Y be the number of accidents in the last 60 miles of the highway. Note that these two intervals of the highway overlap. Compute the covariance $\text{Cov}(X, Y)$ and the correlation coefficient $\rho_{X,Y}$.

Answer: $\text{Cov}(X, Y) = 0.4$ and $\rho_{X,Y} = 1/3$

7. Assume that accidents on a 600 miles long highway occur at a rate of 0.05 per mile.

(a) What is the distribution of the distances between accidents? Write down a formula for the distribution function, give its mean value and variance.

Answer: $W = \text{exponential}(0.05)$,

$$F_W(x) = 1 - e^{-0.05\,x}$$

$$EW = 20, \qquad \operatorname{Var} W = 400$$

(b) Let W be the total number of accidents on the highway. Use normal approximation to a Poisson random variable to compute $P(27 < W < 32)$.

$W = \text{poisson}(0.05 \cdot 600) = \text{poisson}(30) \approx N(30, 30)$

$$P(27 < W < 32) = \Phi\left(\frac{31.5 - 30}{\sqrt{30}}\right) - \Phi\left(\frac{27.5 - 30}{\sqrt{30}}\right) = \Phi(0.27) - \Phi(-0.46) = 0.2836$$

8. A random variable X has distribution function $F(x) = e^x$ for $-\infty < x < 0$. In the problems below, do not forget to indicate the range of the random variable Y.

(a) Find the distribution and density function of $Y = (1 - X)^2$.

Answers: $Y > 1$,

$$F_Y(y) = 1 - e^{1-\sqrt{y}}$$

$$f_Y(y) = \frac{1}{2\sqrt{y}}\, e^{1-\sqrt{y}}$$

(b) [Bonus] Find the distribution and density function of $Y = \frac{1}{X-2}$.

Answers: $-1/2 < Y < 0$,

$$F_Y(y) = 1 - e^{\frac{1}{y}+2}$$

$$f_Y(y) = \frac{1}{y^2}\, e^{\frac{1}{y}+2}$$

9. A random variable X has moment generating function

$$\varphi_X(t) = -\frac{2}{t-2}$$

(a) Find EX.

Answer: $\varphi'(t) = 2/(t-2)^2$, $EX = 1/2$

(b) Find EX^2.

Answer: $\varphi''(t) = -4/(t-2)^3$, $EX^2 = 1/2$

(c) Find $\operatorname{Var} X$. Answer: 1/4.

(d) Find EX^3.

Answer: $\varphi'''(t) = 12/(t-2)^4$, $EX^3 = 3/4$

(e) Determine the type of the random variable X.

Answer: $X =$exponential(2)

(f) [Bonus] Find the skewness and kurtosis of X.