MA 486/586-1C (Statistics), Dr Chernov — Midterm test #1

Show your work. — Fri, Feb 7, 2014

1. (50 pts) A random sample $x_1, \ldots, x_{10}$ from $\mathcal{N}(\mu, \sigma^2)$ yielded

$$\sum_{i=1}^{10} x_i = 20 \qquad \text{and} \qquad \sum_{i=1}^{10} x_i^2 = 67$$

(a) Give an unbiased estimate for μ. Answer: $\hat{\mu} = \bar{x} = 2$.

(b) Construct a 99% confidence interval for μ. Answer: $\bar{x} \pm t_{0.005}(9) \cdot \sqrt{3}/\sqrt{10} = 2 \pm 1.78$.

(c) Give an unbiased estimate for σ^2 Answer: $s^2 = 3$.

(d) Give the maximum likelihood estimate for σ^2. Answer: $\hat{\sigma}^2_{\mathrm{MLE}} = 2.7$.

(e) Construct a 95% confidence interval for σ^2 using χ^2 percentiles from Table IV.

Answer: $\frac{9 \cdot 3}{19.02} < \sigma^2 < \frac{9 \cdot 3}{2.7}$, hence $1.42 < \sigma^2 < 10$.

(Bonus) Construct the shortest 95% confidence interval for σ^2.

Answer: $\frac{9 \cdot 3}{22.912} < \sigma^2 < \frac{9 \cdot 3}{3.187}$, hence $1.18 < \sigma^2 < 8.47$.

2. (20 pts) The following are changes in the price of a stock over 12 consecutive days:

$$-3 \quad 12 \quad -8 \quad 5 \quad 9 \quad 0 \quad -1 \quad 8 \quad 4 \quad -8 \quad 4 \quad 2$$

(a) Determine the mode, the median, the quartiles and the IQR.
Mode: -8 and 4; median: $\tilde{m} = 3$; quartiles: $\tilde{q}_1 = -2$ and $\tilde{q}_3 = 6.5$; IQR is 8.5.
(b) Find sample mean, sample variance and sample standard deviation.
$\bar{x} = 2$, $s^2 = 40$, and $s = \sqrt{40} = 6.32$.
(Bonus) Draw a box-and-whisker diagram.

3. (20 pts) Let $x_1, \ldots, x_n$ be a random sample from the distribution with probability density function

$$f(x;\theta) = \frac{\theta}{x^2}\, e^{-\frac{\theta}{x}} \qquad \text{for} \quad x > 0$$

where $\theta > 0$ is an unknown parameter. Find the maximum likelihood estimate for θ.

Solution:

Likelihood function:

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta}{x_i^2}\, e^{-\frac{\theta}{x_i}} = \frac{\theta^n}{\prod_{i=1}^{n} x_i^2}\, e^{-\theta \sum_{i=1}^{n} \frac{1}{x_i}}$$

Log-likelihood function:

$$\ln L(\theta) = n \ln \theta - \ln \prod_{i=1}^{n} x_i^2 - \theta \sum_{i=1}^{n} \frac{1}{x_i}$$

Derivative of the log-likelihood function:

$$\frac{d}{d\theta} \ln L(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} \frac{1}{x_i}$$

MLE equation:

$$\frac{d}{d\theta} \ln L(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} \frac{1}{x_i} = 0$$

Solving it gives

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

(this is called the **harmonic mean** of the n given numbers $x_1, \ldots, x_n$).

4. (10 pts) Let $x_1, \ldots, x_{10}$ be a sample from a normal random variable $\mathcal{N}(\mu_X, \sigma_X^2)$ and $y_1, \ldots, y_8$ be a sample from a normal random variable $\mathcal{N}(\mu_Y, \sigma_Y^2)$. Their sample means are $\bar{x} = 3$ and $\bar{y} = -1$, and their sample variances are $s_x^2 = 20$ and $s_y^2 = 16$, respectively.

(a) Construct a 90% confidence interval for σ_X^2/σ_Y^2.

Answer:

$$\frac{20}{16} \cdot \frac{1}{3.58} < \frac{\sigma_X^2}{\sigma_Y^2} < \frac{20}{16} \cdot 3.29$$

hence

$$0.34 < \frac{\sigma_X^2}{\sigma_Y^2} < 4.11$$

(Bonus) Construct an 80% confidence interval for $\mu_X - \mu_Y$.

Answer: we use Welch's formula. The number of degrees of freedom is

$$r = \frac{\left(\frac{20}{10} + \frac{16}{8}\right)^2}{\frac{1}{9}\left(\frac{20}{10}\right)^2 + \frac{1}{7}\left(\frac{16}{8}\right)^2} = 15.75$$

We round it **down** to $r = 15$. Now the CI is

$$3 - (-1) \pm t_{0.1}(15)\sqrt{\frac{20}{10} + \frac{16}{8}} = 4 \pm 1.341 \cdot 2 = 4 \pm 2.682$$

hence the CI is $[1.32, 6.68]$.