MA 486-25 (Statistics), Dr Chernov — Midterm test #2

Show your work. — Thu, May 3, 2001

1. (8 pts) Let $x_1, \ldots, x_8$ be taken from $N(\mu, \sigma^2)$ to test $H_0 : \sigma = 3$ against $H_1 : \sigma < 3$. The test results are $\bar{x} = 6$ and $s^2 = 10.2$.

(a) Find the critical region.
(b) Do we accept H_0 at the 10% significance level?
(c) [Bonus] What can you say about the p-value of the test?

Solution:

$$\chi^2 = \frac{7 \cdot 10.2}{9} = 7.93$$

Critical region is

$$\chi^2 < \chi^2_{0.9}(7) = 2.833$$

We accept H_0.

The p-value is in $(0.1, 0.9)$.

2. (12 pts) Let p_1 be the proportion of students who took the precalculus course and then failed in calculus. Let p_0 be the proportion of students who did not take the precalculus course and then failed in calculus. In a random sample of 100 students who took the precalculus course, 15 failed in calculus. In a random sample of 200 students who did not take the precalculus course, 21 failed in calculus. You need to test the hypothesis $H_0 : p_0 = p_1$ at 20% level.
(a) Specify the alternative, in a reasonable way (justify!).
(b) Find the critical region.
(c) Which hypothesis do you accept?
(d) [Bonus] By using normal approximation, find the p-value of the test.

Solution:

The alternative can be $p_0 \neq p_1$ or $p_0 > p_1$ or $p_0 < p_1$ based on any considerations, of any sort. But your considerations should not be based on the experimental data. The alternative hypothesis must be stated *before* the experiment is completed.

Let us assume $H_0 :\ p_0 \neq p_1$.
We have $\hat{p}_0 = 21/200 = 0.105$, $\hat{p}_1 = 15/100 = 0.15$,

$$\hat{p} = \frac{21 + 15}{200 + 100} = 0.12$$

$$Z = \frac{\hat{p}_0 - \hat{p}_1}{\sqrt{\hat{p}(1 - \hat{p})(1/n_1 + 1/n_2)}} = -1.13$$

Critical region:

$$|Z| > z_{\alpha/2} = 1.282$$

We accept H_0.

The p-value is $= 2(1 - \Phi(|Z|)) = 0.2584$

3. (10 pts) In a school fund-raising lottery, each ticket wins one of four prizes, call them A, B, C, D, which are supposed to occur in the ratio 1:2:5:10. In one day, 90 students have bought tickets (one each). Out of those students, 3 won prize A, 7 won prize B, 20 won prize C, and the rest – D. Are these data consistent with the announced ratio of prizes? Let $\alpha = 0.05$. What can you say about the p-value of the test?

Solution:

Probabilities of classes are 1/18, 2/18, 5/18, and 10/18.

Classes	A	B	C	D
Theory ($= np_i$)	5	10	25	50
Experiment	3	7	20	60

The test statistic is

$$Q = \frac{(3-5)^2}{5} + \frac{(7-10)^2}{10} + \frac{(20-25)^2}{25} + \frac{(60-50)^2}{50} = 4.7$$

The critical region is

$$Q > \chi^2_{0.05}(3) = 7.815$$

We accept H_0.

4. (12 pts) Let $x_1, \ldots, x_{16}$ be taken from $N(\mu, 25)$. To test $H_0 : \mu = 10$ against $H_1 : \mu > 10$ let the critical region be $\bar{x} \geq 12.4$.
(a) Write down formulas for $\alpha, \beta, K(\mu)$.
(b) Compute α and $K(11)$, $K(12)$, $K(12.5)$, $K(13)$, $K(14)$. Sketch the graph of the power function.
(c) Let $\bar{x} = 12.2$. What is the p-value of the test?

Solution:

$$\alpha = 1 - \Phi\left(\frac{12.4 - 10}{\sqrt{25}/\sqrt{16}}\right) = 1 - \Phi(1.92) = 0.0274$$

$$\beta = 1 - K(\mu)$$

$$K(\mu) = 1 - \Phi\left(\frac{12.4 - \mu}{\sqrt{25}/\sqrt{16}}\right)$$

$$K(11) = 1 - \Phi(1.12) = 0.1314$$

$$K(12) = 1 - \Phi(0.32) = 0.3745$$

$$K(12.5) = 1 - \Phi(-0.08) = 0.5319$$

$$K(13) = 1 - \Phi(-0.48) = 0.6844$$

$$K(14) = 1 - \Phi(-1.28) = 0.8997$$

The p-value is

$$1 - \Phi\left(\frac{12.2 - 10}{\sqrt{25}/\sqrt{16}}\right) = 1 - \Phi(1.76) = 0.0392$$

5. (8 pts) It is claimed that 40% of doctors recommend a certain brand of pain relief medicine. A consumer group doubts this claim (they think the estimate is incorrect, either way). It conducts a survey that shows that out of 600 doctors, 213 prescribe this brand of medicine.
(a) State the null hypothesis H_0 and the alternative hypothesis H_1.
(b) Find the critical region for the significance level $\alpha = 0.02$.
(c) Which hypothesis do you accept?
(d) [Bonus] By using normal approximation, find the p-value of the test.

Solutions:

$$H_0: \ p = 0.4 \quad H_1: \ p \neq 0.4$$

$$Z = \frac{y/n - p_0}{\sqrt{p_0(1-p_0)/n}} = -2.25$$

Critical region:

$$|Z| > z_{\alpha/2} = 2.326$$

We accept H_0.

p-value is $2(1 - \Phi(2.25)) = 0.0244$

6. (10 pts) Let $x_1, \ldots, x_{28}$ be taken from $N(\mu, \sigma^2)$ to test $H_0 : \mu = 2$ against $H_1 : \mu > 2$. The test results are $\sum x_i = 71$ and $\sum x_i^2 = 207$.

(a) Find the critical region.
(b) Do we accept H_0 at the 1% significance level?
(c) What is the approximate p-value of the test?

Solution:

First of all, $\bar{x} = 71/28$ and $s^2 = (207 - 71^2/28)/27 = 0.9987$

$$T = \frac{71/28 - 2}{\sqrt{0.9987}/\sqrt{28}} = 2.8366$$

Critical region is

$$T > t_{0.01}(27) = 2.473$$

We accept H_1.

The p-value is < 0.005.

7. (12 pts) Let X be $N(\mu, 9)$. We need to test the hypothesis $H_0 : \mu_0 = -2$ against $H_1 : \mu_1 = -3.5$. Let the critical region be $\bar{x} \leq c$. Find values for n and c so that $\alpha = 0.01$ and $K(\mu_1) = 0.9975$. Do not forget to find the values of *both* n and c.

Solution:

$$\frac{c - (-2)}{3/\sqrt{n}} = -z_{0.01} = -2.326$$

$$\frac{c - (-3.5)}{3/\sqrt{n}} = z_{0.0025} = 2.807$$

$$n = \frac{(2.326 + 2.807)^2 \cdot 9}{(-3.5 + 2)^2} = 105.4$$

so $n = 106$, and

$$c = -2.677$$

8. (14 pts) Let X denote the number of traffic accidents in a small town on one day. The following 20 observations of X (i.e., the number of recorder accidents on 20 consecutive days):

3	1	4	3	3	2	5	3	0	2
6	7	4	1	2	6	0	3	4	1

Test the hypothesis that these numbers come from a Poisson distribution. Let α =5%. Remember to estimate the parameter. When grouping data, apply the "minimal safety requirement" $np_i \geq 1.5$.

Solution:

$\hat{\lambda} = \bar{x} = 60/20 = 3$. Next we use the formula

$$p_k = P(X = k) = \frac{\hat{\lambda}^k}{k!}\, e^{-\hat{\lambda}}$$

Classes	0,1	2	3	4	5	$6, 7, \ldots$
Theory $(= np_i)$	3.96	4.48	4.48	3.36	2.02	1.68
Experiment	5	3	5	3	1	3

The test statistic is

$$Q = \frac{(5 - 3.96)^2}{3.96} + \cdots + \frac{(3 - 1.68)^2}{1.68} = 2.40$$

The critical region is

$$Q > \chi^2_{0.05}(4) = 9.488$$

We accept H_0.

9. (14 pts) Let $x_1, \ldots, x_{31}$ and $y_1, \ldots, y_{25}$ be two independent samples taken from $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$, respectively. The test results are $\bar{x} = 8$, $s_x^2 = 120$, $\bar{y} = 10$ and $s_y^2 = 160$.

Test the hypothesis that these two normal distributions are equal. Let $\alpha = 5\%$. Remember that there are two parameters to take care of. Find explicitly the critical regions.

Solution:

First, we test the hypothesis $H_0 : \sigma_x^2 = \sigma_y^2$ against $H_1 : \sigma_x^2 \neq \sigma_y^2$. The critical region is

$$s_x^2/s_y^2 = 0.75 > F_{0.025}(30, 24) = 2.21 \quad \text{or} \quad s_y^2/s_x^2 = 1.33 > F_{0.025}(24, 30) = 2.14$$

We accept H_0.

Then, we test the hypothesis $H_0 : \mu_x^2 = \mu_y^2$ against $H_1 : \mu_x^2 \neq \mu_y^2$. The test statistic is

$$T = \frac{8 - 10}{\sqrt{\frac{30\cdot120+24\cdot160}{54}}\sqrt{\frac{1}{31}+\frac{1}{25}}} = -0.634$$

The critical region is

$$|T| > t_{0.025}(54) = 1.96$$

We accept H_0.