MA 486/586-2D (Statistics), Dr Chernov — Midterm test #2

Show your work. — Thu, Mar 9, 2006

1. (25 pts) Let $x_1, \dots, x_n$ be a random sample from the distribution with probability density function

$$f(x;\theta) = \frac{\sqrt{\theta}}{\sqrt{\pi}}\, e^{-\theta x^2} \quad \text{for} \quad -\infty < x < \infty$$

where $\theta > 0$ is an unknown parameter.

(a) Find sufficient statistics.

Answer: $u = \sum x_i^2$.

(b) Find the maximum likelihood estimate for θ, and make sure that the MLE is a function of sufficient statistics.

Answer: $\hat{\theta} = \frac{n}{2u}$.

(c) Find the Rao-Cramer lower bound on the variance of unbiased estimates of θ.

Answer: $\sigma_n^2 = \frac{2\theta^2}{n}$.

(d) What is the asymptotic distribution of the MLE?

Answer: normal $N(\theta, \sigma_n^2)$.

2. (15 pts) Let X be the number of female children in a three-child family. We shall test the hypothesis that X is $b(3, 0.5)$.
(a) Compute the corresponding probabilities for the values $X = 0, 1, 2, 3$.
(b) Suppose a random sample of 56 families, with three children each, contained 6 families with no daughters, 22 families with one daughter, 19 families with two daughters, and 9 families with three daughters. Test the hypothesis at 5% significance level.
(Bonus) What is an approximate p-value of the test? Interpret it.

Answer: $Q = 20/21 = 0.9524$. Critical region is $Q > \chi^2_{0.05}(3) = 7.815$. We accept H_0. The p-value is between 0.1 and 0.9 (closer to 0.9).

3. (15 pts) In a year, basketball player A attempted 100 free throws and made 74 of them; basketball player B attempted 80 free throws and made 61 of them.

(a) Test the hypothesis that the players A and B make the same proportion of free throws, on the average, at 20% level.

Answer: $\hat{p} = 3/4$, $Z = -0.3464$. Critical region is $|Z| > z_{0.1} = 1.282$. We accept H_0.

(b) Find the p-value of the test.

Answer: p-value is $2[1 - \Phi(0.35)] = 0.7264$.

4. (20 pts) Let $x_1, \dots, x_{25}$ and $y_1, \dots, y_{31}$ be two independent samples taken from $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$, respectively. The test results are $\bar{x} = 11$, $s_x^2 = 20$, $\bar{y} = 14$ and $s_y^2 = 25$.
(a) Test the hypothesis of equality of these two normal random variables at the level $\alpha = 5\%$.

Answer: Step 1: $F = 1.25$, critical region is $F > F_{0.025}(30, 24) = 2.21$ and $F < 1/F_{0.025}(24, 30) = 1/2.14 = 0.467$. We accept $H_0\colon \sigma_X^2 = \sigma_Y^2$.

Step 2: $T = -2.338$, critical region is $|T| > t_{0.025}(54) \approx z_{0.025} = 1.960$. We reject H_0, so the normal random variables are different.

(Bonus) What can yo say about the p-value of the test?

Answer: p-value is $2[1 - \Phi(2.338)] = 0.0192$.

5. (25 pts) Let $x_1, \ldots, x_{100}$ be a sample taken from $N(\mu, 16)$. We shall test the hypothesis $H_0 : \mu = 5$ against $H_1 : \mu < 5$.

(a) Let the critical region be $\bar{x} < 4.1$. Find α and write down formulas for β and $K(\mu)$.

Answer: $\alpha = \Phi(-2.25) = 0.0122$, $\beta = 1 - \Phi\left(\frac{4.1-\mu}{0.4}\right)$, $K(\mu) = 1 - \beta$.

(b) Compute $K(3.3)$, $K(3.7)$, $K(4.1)$, $K(4.5)$ and $K(4.9)$. Sketch the graph of the power function.

Answer: $K(3.3) = 0.9772$, $K(3.7) = 0.8413$, $K(4.1) = 0.5$, $K(4.5) = 0.1587$, $K(4.9) = 0.0228$.

(c) Assume that the sample yields $\bar{x} = 4.08$. What is the p-value of the test?

Answer: p-value is $1 - \Phi(2.3) = 0.0107$.

(d) Suppose we want $\alpha = 0.002$ and $K(4.2) = 0.996$. How large a sample will be necessary? What would be the critical region?

Answer: $n \geq 765$, $c = 4.5834$. Critical region is $\bar{x} < c$.