MA 587/687 (Advanced Probability), Dr. Chernov — Final Exam

10 problems. — April 2013

587 students: do 8 problems for full credit. (If you do more, you get extra points.)
687 students: do 9 problems for full credit. (If you do more, you get extra points.)

1. Let X and Y be continuous random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} cy & \text{for } x^2 \le y \le \sqrt{x}, \quad 0 \le x \le 1, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant c. Answer: $c = \frac{20}{3}$
(b) Find the marginal density function of X. $f_X(x) = \frac{10}{3}(x - x^4)$ for $0 < x < 1$
(c) Find the marginal density function of Y. $f_Y(y) = \frac{20}{3}(y^{3/2} - y^3)$ for $0 < y < 1$

2. Let X and Y be independent random variables with density functions

$$f_X(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1, \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2 & \text{for } -1 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

Find the density function for $X+Y$ by using the convolution formula. Answer:

$$f_{X+Y}(a) = \begin{cases} \frac{1}{4}a^4 + a + \frac{3}{4} & \text{for } -1 < a < 0 \\ \frac{3}{2}a^2 - 2a + \frac{3}{4} & \text{for } 0 < a < 1 \\ -\frac{1}{4}a^4 + \frac{3}{2}a^2 - a & \text{for } 1 < a < 2 \\ 0 & \text{otherwise} \end{cases}$$

3. The conditional distribution of Y, given X, is uniform on the interval $[-X, X]$. The marginal density of X is

$$f_X(x) = \begin{cases} x^{-2} & \text{for } x \ge 1, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the joint density function $f_{XY}(x,y)$. $f_{XY}(x,y) = \frac{1}{2x^3}$ for $x > 1$, $-x < y < x$
(b) Draw the region where $f_{XY} > 0$.
(c) Find the conditional density of X, given $Y = y$. Can you name the corresponding distribution? In each interval, it is a Pareto distribution

$$f_{X|Y}(x|y) = \begin{cases} \frac{2y^2}{x^3} & \text{for } y < -1 \text{ and } y > 1 \\ \frac{2}{x^3} & \text{for } -1 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(d) Find the conditional expectation of X, given $Y = y$. Answer:

$$\mathbb{E}(X|Y) = \begin{cases} -2y & \text{for } y < -1 \\ 2y & \text{for } y > 1 \\ 2 & \text{for } -1 < y < 1 \end{cases}$$

4. A machine consists of two components, whose lifetimes have the joint density function

$$f(x, y) = \begin{cases} \frac{24}{7}xy & \text{for } \ x > 0, \ y > 0, \ x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The machine operates until both components fail. Find the expected operational time of the machine. Answer: $\mathbb{E}(T) = \frac{11}{140}$

5. Independent random variables X, Y and Z are identically distributed. Let $W = X+Y$. The moment generating function of W is $M_W(t) = (0.2 + 0.5e^{-t} + 0.3e^t)^2$.
(a) Find the moment generating function of $V = X+Y+Z$. $M_V(t) = (0.2+0.5e^{-t}+0.3e^t)^3$
(b) Find all possible values of X and the respective probabilities. values: $0, -1, 1$; respective probabilities: $0.2, 0.5, 0.3$
(c) Find all possible values of W and the respective probabilities. values: $0, -1, 1, -2, 2$; respective probabilities: $0.34, 0.2, 0.12, 0.25, 0.09$

6. Let $X_1, \dots, X_n$ be independent and identically distributed random variables, each of them is uniform on the interval $[0, n]$. Let Y_n be the minimum of $X_1, \dots, X_n$, i.e., $Y_n = \min\{X_1, \dots, X_n\}$. .
(a) Find the cumulative distribution $F_{Y_n}(x)$ of Y_n. $F_{Y_n}(x) = 1 - (1 - \frac{x}{n})^n$ for $0 < x < n$
(b) Find the limit of $F_{Y_n}(x)$ as $n \to \infty$. $1 - e^{-x}$ for $x > 0$
(c) Is it true that Y_n converges in probability to 0? In other words, is it true that for arbitrary $\varepsilon > 0$

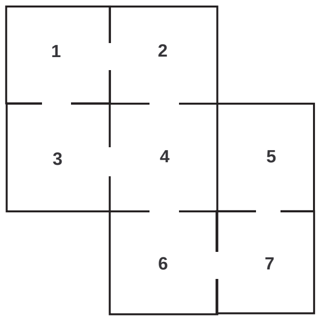
$$\lim_{n\to\infty} \mathbb{P}(|Y_n - 0| \le \varepsilon) = 1.$$

No, not true. This limit is $1 - e^{-\varepsilon}$

7. Let X and Y be the number of hours that a randomly selected person watches movies and sporting events, respectively, during a three-month period. The following information is known about X and Y:

$$\begin{aligned} \mathbb{E}(X) &= 25 \\ \mathbb{E}(Y) &= 35 \\ \sigma_X &= 4 \\ \sigma_Y &= 5 \\ \rho_{X,Y} &= -0.125 \end{aligned}$$

A hundred people are randomly selected and observed for these three months. Let T be the total number of hours that these hundred people watch movies or sporting events during this three-month period. Approximate the value of $\mathbb{P}(5900 < T < 6150)$. Answer: 0.9463



8. A rat runs through the maze shown in the above figure. At each step it leaves the room it is in by choosing at random one of the doors out of the room.
(a) Give the transition matrix P for this Markov chain.
(b) Draw the corresponding graph.
(c) Is it an irreducible (i.e., ergodic) chain? Explain.
(d) Is it a regular (i.e., irreducible and aperiodic) chain? Explain. No. There are two groups of states, $\{1, 4, 7\}$ and $\{2, 3, 5, 6\}$, that repeat periodically.
(e) Find the fixed (i.e., stationary) probability vector. Answer: $\mathbf{w} = (\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{3}{14}, \frac{1}{14}, \frac{1}{7}, \frac{1}{7})$
(f) If v is an probability vector, is it true that vP^n converges to the stationary vector, as $n \to \infty$? Does it converge in some special sense? What type of convergence takes place? No, it does not converge directly, but it converges in the Cesaro sense.

9. An absorbing Markov chain has the following transition matrix:

$$P = \begin{bmatrix} 0.4 & 0.15 & 0.2 & 0.25 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix}$$

(a) Draw the corresponding graph, identify transient and absorbing states.
(b) Find the fundamental matrix $\mathbf{F}$. $\mathbf{F} = \begin{bmatrix} 2 & 1 \\ 0.8 & 2.4 \end{bmatrix}$
(c) Find the product $\mathbf{FR}$ and interpret its values. $\mathbf{FR} = \begin{bmatrix} 0.5 & 0.5 \\ 0.6 & 0.4 \end{bmatrix}$
(d) Compute the product $\mathbf{Fu}$, where $\mathbf{u}$ is a vectors of ones, and interpret its values. $\mathbf{Fu} = \begin{bmatrix} 3 \\ 3.2 \end{bmatrix}$

(e) How many stationary (equilibrium) vectors does this chain have? Describe all. Answer: infinitely many, all of the form $\mathbf{w} = (0, p, q, 0)$ with $p \geq 0, q \geq 0, p + q = 1$

10. Let X and Y have a bivariate normal distribution. Suppose X has mean 2 and variance 16, and Y has mean -2 and variance 9. Let the correlation between X and Y be -0.6.

(a) Write down a formula for the joint density function Answer:

$$f(x, y) = \frac{1}{19.2\pi} e^{-\frac{1}{2}\left[\left(\frac{x-2}{4}\right)^2 + \left(\frac{y+2}{3}\right)^2 + 1.2\left(\frac{x-2}{4}\right)\left(\frac{y+2}{3}\right)\right]}$$

(b) Find the covariance matrix $\mathbf{V}$ and its inverse $\mathbf{V}^{-1}$.

$$\mathbf{V} = \begin{bmatrix} 16 & -7.2 \\ -7.2 & 9 \end{bmatrix}, \qquad \mathbf{V}^{-1} = \begin{bmatrix} 0.098 & 0.078 \\ 0.078 & 0.174 \end{bmatrix}$$

(d) Compute the probability $\mathbb{P}(|X| < 3)$. Answer: 0.4931

(e) Compute the conditional probability $\mathbb{P}(|X| < 3 | Y = -1)$. Answer: 0.6172

(f) Find a constant c such that the variable $U = X + cY$ is independent of Y. $c = 0.8$