MA 631-22 (Applied linear algebra), Dr. Chernov — Midterm test

Justify your work. — Thur, October 22

1. (4 pts) An $n \times n$ matrix A is said to be skew-symmetric if $A = -A^t$.
(i) Show that if n is odd, then $\det A = 0$.
(ii) Find a 2×2 skew-symmetric matrix A with $\det A \neq 0$.

2. (5 pts) Let $B = \{(1,1,1), (2,1,1,), (4,1,0)\}$ be a basis in $\mathbb{R}^3$. Find the transition matrices $P_{B,E}$ and $P_{E,B}$, where E stands for the standard basis in $\mathbb{R}^3$. Find a matrix A such that $Av = (v)_B$ for all $v \in \mathbb{R}^3$.

3. (4 pts) (i) Show that the matrices $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $b \neq 0$ and $B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ are *not* similar.
(ii) Based on this, prove that the matrix A is not diagonalizable over the complex field.

4. (3 pts) (i) Show that if a square matrix A is diagonalizable, then so is A^2.
(ii) Find a 2×2 matrix A such that A^2 is diagonalizable but A is not. (Use problem 3.)

5. (5 pts) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $(Tp)(x) = p(x) + x^2 p(x^{-1})$.
(i) Compute $[T]_B$ for the basis $B = \{1, x, x^2\}$ of $P_2(\mathbb{R})$.
(ii) Find all the eigenvalues of T.
(iii) Is T diagonalizable?

6. (4 pts) Let A be a square invertible matrix with eigenvalues $\lambda_1, \ldots, \lambda_s$. Compute $C_{A^{-1}}(x)$ in terms of $C_A(x)$. Show that the matrix A^{-1} has eigenvalues $\lambda_1^{-1}, \ldots, \lambda_s^{-1}$. For an extra credit (3pts), prove that for each $i = 1, \ldots, s$ the algebraic and geometric multiplicities of λ_i for A are equal to the algebraic and geometric multiplicities of λ_i^{-1} for A^{-1}, respectively.

7. (5 pts) Let a square matrix A be diagonalizable, and its characteristic polynomial be $C_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Prove that

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

where 0 is the zero matrix. (Hint: First assume that A is actually diagonal.)

8. [Bonus problem, JPE 1994] (5 pts) Recall that a square matrix A is diagonalizable iff $X^{-1}AX$ is a diagonal matrix, for some matrix X. Show that the columns of the matrix X are the eigenvectors of the matrix A.

9. [Bonus problem] (5 pts) Consider the linear transformation $T : C[0,1] \to C[0,1]$ defined by

$$f(x) \overset{T}{\mapsto} g(x) = \int_0^x f(t)\,dt$$

Prove that T has no eigenvalues.