MA 631-22 (Applied linear algebra), Dr. Chernov | Final exam
Justify your work. | Tue, November 24

1. Find a Jordan canonical form for

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 2 & -1 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

2. Find a Jordan canonical form and a Jordan basis for $A = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$

3. (a) Let A be a 12×12 complex matrix with $C_A(x) = x^7(x-i)^5$, $M_A(x) = x^3(x-i)^2$, and $\dim E_0 = 4$, $\dim E_i = 3$. Find a Jordan canonical form of A.
(b) Let A be a 10×10 complex matrix with $C_A(x) = (x^2+1)^5$, $\dim E_i = 1$ and $\dim E_{-i} = 4$. Find the minimal polynomial of A.

3. Let $W := \text{span}\{(2,-1,2,-1), (1,3,1,3), (0,2,0,2)\}$ in $\mathbb{R}^4$.
(a) Find $\dim W$ and a basis of W.
(b) Find a basis of $W^\perp$ in $\mathbb{R}^4$ with the standard inner product.

4. Let $v_1 = (1,1,0)$, $v_2 = (2,2,2)$, $v_3 = (0,2,1)$ in $\mathbb{R}^3$.
(a) Find an ONB $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$ by using the Gram-Schmidt procedure.
(b) Find c_1, c_2, c_3 such that $c_1u_1 + c_2u_2 + c_3u_3 = (0,2,2)$.

5. Let $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ for some $\theta \in [0, 2\pi)$. Show that A is orthogonal and $\det A = -1$. Prove that $\lambda = \pm 1$ are the eigenvalues of the above matrix A, find the corresponding eigenvectors. For an extra credit, show that if A is a real 2×2 orthogonal matrix with $\det A = -1$, then A has the above form.

6. Let V be a finite dimensional inner product space, and W a subspace of V. Then $V = W \oplus W^\perp$, that is, each $v \in V$ is uniquely expressed in the form $v = v_1 + v_2$ with $v_1 \in W$ and $v_2 \in W^\perp$. Define a linear operator T by $Tv = v_2 - v_1$.
(i) Prove that T is an isometry.
(ii) Assume that $\dim W = 1$. Prove that $\det T = -1$.
(iii) Let $V = \mathbb{R}^3$ and $W = \text{span}\{(1,1,0)\}$. Find the matrix representation of T in the canonical basis $\{e_1, e_2, e_3\}$.

7. Let $\{w_1, \ldots, w_n\}$ be an ONB in $\mathbb{R}^n$. What is $||w_1 + \cdots + w_n||$? Assuming that n is even, what is $||w_1 - w_2 + w_3 - \cdots - w_n||$?

continued on the back

8. [Extra credit] Let u be a unit vector in $\mathbb{R}^n$. Prove that $A = I - 2uu^t$ is an orthogonal matrix. Then show that $Au = -u$ and $Av = v$ for any vector v orthogonal to u. Is there any connection with Problem 6?