MA 632-14 (Applied linear algebra), Dr. Chernov | Midterm test

Justify your work. | Wed, February 11

1. (a) Let $A \in \mathbb{R}^{n\times n}$ be orthogonal, symmetric and positive definite. Show that $A = I$.
(b) Let $A \in \mathbb{C}^{n\times n}$ be unitary, Hermitean and positive definite. Show that $A = I$.

2. Let $A \in \mathbb{C}^{n\times n}$. Assume that $A^*A = AA^*$ and A is upper-triangular. Show that A is diagonal.

3. Let $R_1, R_2 \in \mathbb{R}^{2\times 2}$ be two reflection matrices. Prove that R_1R_2 is a rotation matrix, i.e. $R_1R_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in [0, 2\pi)$.

4. Find the condition numbers $\kappa_1(A)$, $\kappa_2(A)$ and $\kappa_\infty(A)$ for

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

5. Find a QR-factorization for

$$A = \begin{pmatrix} -4 & -2 \\ 3 & 5 \end{pmatrix}$$

6. Let A be a 3×3 symmetric positive definite real matrix. The Cholesky decomposition says that there is a unique lower triangular matrix G with positive diagonal elements such that $A = GG^t$. How many lower triangular real matrices G_i exist that satisfy the equation $A = G_iG_i^t$? [Hint: use the uniqueness of LDL^t-decomposition.]

7. Let data points $\{(x_i, y_i) \in \mathbb{R}^2 : 1 \le i \le m\}$ be given. Find explicit formulas for the least square solution of the system $ax_i + b = y_i$, $1 \le i \le m$. That is, find a and b in terms of x_i, y_i. Use convenient Gauss brackets: $[X^k] = x_1^k + \cdots + x_m^k$ and $[X^kY] = x_1^k y_1 + \cdots + x_m^k y_m$.

8. Let A be a 3×3 orthogonal real matrix, and $\det A = -1$. Show that there is an orthogonal matrix Q such that

$$Q^{-1}AQ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in [0, 2\pi)$.