MA 632-22 (Applied linear algebra), Dr. Chernov — Midterm test

Justify your work. Do 6 out of 7 problem for full credit. — Thu, February 11

1. (a) Let $A \in \mathbb{R}^{n\times n}$ be orthogonal, symmetric and positive definite. Show that $A = I$.
(b) Let $A \in \mathbb{C}^{n\times n}$ be unitary, Hermitean and positive definite. Show that $A = I$.
(c) Let $A \in \mathbb{R}^{m\times n}$, $m < n$. Show that $A^t A$ is not positive definite.

2. (a) Show that $\kappa(A) = \kappa(A^{-1})$.
(b) Show that $\kappa(AB) \le \kappa(A)\kappa(B)$.
(c) Show that $\kappa_\infty(A) = \kappa_1(A^t)$.

3. Find the condition numbers $\kappa_1(A)$, $\kappa_2(A)$ and $\kappa_\infty(A)$ for

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}$$

4. (JPE September 1998) Let $Ax = b$ and $r = b - Ax_c$ for some nonsingular matrix $A \in \mathbb{R}^{n\times n}$. Prove that

$$\frac{1}{\kappa(A)} \frac{||r||}{||b||} \le \frac{||x - x_c||}{||x||} \le \kappa(A) \frac{||r||}{||b||}$$

Comment on the significance of this inequality for $\kappa(A)$ close to 1 and for $\kappa(A)$ large.

5. Find a QR-decomposition for

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

6. Prove that the matrix

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

is positive definite in two ways. First, use Sylvester's theorem. Second, find the Cholesky factorization for A.

7. Let $R_1, R_2 \in \mathbb{R}^{2\times 2}$ be two reflection matrices. Prove that $R_1 R_2$ is a rotation matrix, i.e. $R_1 R_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in [0, 2\pi)$.