MA 632-14 (Applied linear algebra), Dr. Chernov — Final exam

Justify your work. — Mon, March 16

1. Let $A \in \mathbb{R}^{n\times n}$ be invertible, $Ax = b$ and $A(x + \Delta x) = b + \Delta b$. Prove directly (i.e. without reference to a more general formula proven in class) that

$$\frac{||\Delta x||}{||x||} \leq \kappa(A)\frac{||\Delta b||}{||b||}$$

2. Let $A \in \mathbb{R}^{m\times n}$, $m > n$, $\text{rank}(A) = n$, $A = QR$ the QR-decomposition of A, and $b \in \mathbb{R}^n$. Show that the least squares solution of $Ax = b$ can be expressed in terms of Q, R and b.

3. Find the numerical rank with tolerance 0.9 of the matrix

$$A = \begin{pmatrix} 3 & 2 \\ -4 & -5 \end{pmatrix}$$

4. Let $A \in \mathbb{C}^{n\times n}$ be Hermitean and λ_1 the smallest eigenvalue of A. Show that $\lambda_1 \leq \min_i a_{ii}$.

5. Let n be an arbitrary positive integer and $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be defined by $a_{ij} = 11^{-|i-j|}$ for $i, j = 1, \ldots, n$. Show that $\sigma(A) \subset (0.8, 1.2)$.

6. Choose an initial vector and calculate three iterations of the power method (Rayleigh quotient method) to approximate the largest eigenvalue of

$$A = \begin{pmatrix} -2 & 6 \\ 3 & 5 \end{pmatrix}$$

7. Find the condition numbers of the eigenvalues of the matrix in the previous problem.