MA 632-22 (Applied linear algebra), Dr. Chernov — Final exam

Justify your work. — Thur, March 11

Full credit is given for 8 problems out of 9. Do as much as you can for extra credit.

1. Let $A \in \mathbb{C}^{n\times n}$ be Hermitean and λ_1 the smallest eigenvalue of A. Show that $\lambda_1 \le \min_i a_{ii}$.

2. Find the numerical rank with tolerance $\varepsilon = 0.3$ of the matrix

$$A = \begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix}$$

3. Let n be an arbitrary positive integer and let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be defined by

$$a_{ij} = 10^{-|i-j|} \qquad \text{for} \quad i,j = 1,\dots,n$$

Show that $\sigma(A) \subset [7/9, 11/9]$.

4. With the initial vector $q_0 = (1,1)^t$, calculate three iterations of the power method with Rayleigh quotient for the matrix

$$A = \begin{pmatrix} -2 & -6 \\ -4 & 3 \end{pmatrix}$$

The choice of the scaling factor σ_k is yours.

5. Find the condition numbers of the eigenvalues of the matrix in the previous problem.

6. (JPE, May 1998) For any matrix $A = (a_{ij})$, $A \in \mathbb{R}^{m\times n}$ define

$$||A||_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}$$

This is the Frobenius matrix norm. Show that if $Q \in \mathbb{R}^{m\times m}$ is orthogonal, then $||QA||_F = ||A||_F$. Then show that

$$||A||_F = (\sigma_1^2 + \cdots + \sigma_r^2)^{1/2}$$

where σ_i are the singular values of A.

7. Use a rotation matrix to compute a QR decomposition for the matrix

$$A = \begin{pmatrix} 3 & 1 & 2 \\ -4 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix}$$

8. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be *skew Hermitean* if $A^* = -A$.

(a) Prove that if A is skew Hermitean and B is unitary equivalent to A, then B is also skew Hermitean.

(b) What special form does the Shur decomposition theorem take for a skew Hermitean matrix A?

(c) Prove that the eigenvalues of a skew Hermitean matrix are purely imaginary, i.e. they satisfy $\bar{\lambda} = -\lambda$.

9. Let $q_0 = (1, 1)^t$ and

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with a real $\lambda \neq 0$. Prove that

$$A^k q_0 = \begin{pmatrix} \lambda^k + k\lambda^{k-1} \\ \lambda^k \end{pmatrix}$$

for all $k \geq 1$. Compute the Rayleigh quotient for the vector $q_k = A^k q_0$, i.e.

$$\rho_k = \frac{q_k^t A q_k}{q_k^t q_k}$$

Prove that $\rho_k \to \lambda$ as $k \to \infty$. What is the speed of convergence, i.e. what can you say about $|\rho_k - \lambda|$ for large k?