MA 645-4A (Real Analysis), Dr. Chernov — Midterm test

Full credit for 5 problems. — Thu, Oct 26, 2006

1. Suppose $f \in L^1_\mu(\mathbb{R})$, where μ is the Lebesgue measure on $\mathbb{R}$. Is it true that $\lim_{x\to\infty} f(x) = 0$? (Prove or give a counterexample.)

Answer: this is false. A counterexample:

$$f(x) = \begin{cases} 1 & \text{if } \ x \in [n, n+1/n^2] \ \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

A more trivial example:

$$f(x) = \begin{cases} 1 & \text{if } \ x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

2. Does there exist a **nonmeasurable** (with respect to the Lebesgue σ-algebra) set $A \subset \mathbb{R}$ such that the set

$$B = \{x \in A \colon x \text{ is irrational}\}$$

is Lebesgue measurable?

Answer: no. Indeed, if B is measurable, then A is measurable, too, because

$$A = B \cup (A \cap \mathbb{Q}).$$

Note: $A \cap \mathbb{Q}$ is a finite or countable set (hence automatically measurable).

3. Let $f\colon [0,1] \to [0,\infty)$. Suppose

$$\lim_{n\to\infty} \int_{[0,1]} f^n \, d\mu$$

exists and is finite. Prove that $\mu\big(\{x\colon f(x) > 1\}\big) = 0$, where μ is the Lebesgue measure.

Proof: let $A = \big(\{x\colon f(x) > 1\}\big) > 0$. Then $\{f^n\}$ is a monotonically increasing sequence of functions on A that converges to infinity. Thus by the Lebesgue monotone convergence theorem

$$\lim_{n\to\infty} \int_A f^n \, d\mu = \int_A \infty \, d\mu = \infty \cdot \mu(A).$$

On the other hand,

$$\lim_{n\to\infty} \int_A f^n \, d\mu \le \lim_{n\to\infty} \int_{[0,1]} f^n \, d\mu < \infty.$$

Thus $\mu(A) = 0$.

4. Let

$$A = \left\{ x \in [0,1] \colon \left| x - \frac{p}{q} \right| < \frac{1}{q^3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Show that $\mu(A) = 0$, where μ is the Lebesgue measure on $\mathbb{R}$. (Hint: use the analogue of the Borel-Cantelli lemma.)

Proof: denote

$$A_{p,q} = \left\{ x \in [0,1] \colon \left| x - \frac{p}{q} \right| < \frac{1}{q^3} \right\}.$$

Note that $\mu(A_{p,q}) = 2/q^3$, and $0 \le p \le q$. Thus

$$\sum_{p,q} \mu(A_{p,q}) \le \sum_{q=1}^{\infty} \frac{2(q+1)}{q^3} \le \sum_{q=1}^{\infty} \frac{4}{q^2} < \infty,$$

so the claim follows from the analogue of the Borel-Cantelli lemma.

5. Let $f\colon \mathbb{R} \to [0,\infty)$ and $f_n\colon \mathbb{R} \to [0,\infty)$ for $n = 1, 2, \ldots$. Assume that $f \in L^1$ and $f_n \in L^1$ for every n. Assume that $f_n \to f$ pointwise on $\mathbb{R}$ and

$$\int_{\mathbb{R}} f_n \, d\mu \to \int_{\mathbb{R}} f \, d\mu$$

as $n \to \infty$. Use Fatou's lemma to show that

$$\int_E f_n \, d\mu \to \int_E f \, d\mu$$

for every Lebesgue measurable set $E \subset \mathbb{R}$.

Proof. By Fatou's lemma

$$\liminf_{n\to\infty} \int_E f_n \, d\mu \geq \int_E \liminf_{n\to\infty} f_n \, d\mu = \int_E f \, d\mu$$

Since

$$\int_{E^c} f_n \, d\mu = \int_{\mathbb{R}} f_n \, d\mu - \int_E f_n \, d\mu,$$

we conclude that

$$\limsup_{n\to\infty} \int_{E^c} f_n \, d\mu \leq \int_{\mathbb{R}} f \, d\mu - \int_E f \, d\mu = \int_{E^c} f \, d\mu.$$

On the other hand, by Fatou's lemma applied to E^c

$$\liminf_{n\to\infty} \int_{E^c} f_n \, d\mu \geq \int_{E^c} \liminf_{n\to\infty} f_n \, d\mu = \int_{E^c} f \, d\mu.$$

This implies

$$\lim_{n\to\infty} \int_{E^c} f_n \, d\mu = \int_{E^c} f \, d\mu$$

for any measurable $E \subset \mathbb{R}$.

6. Construct a sequence of functions $f_n\colon [0,1] \to [0,1]$ such that

$$\lim_{n\to\infty} \int_{[0,1]} f_n \, d\mu = 0$$

(here μ is the Lebesgue measure), and yet the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges for **no** $x \in [0,1]$.

For $n \in \mathbb{N}$, denote

$$S_n = \sum_{i=1}^{n} \frac{1}{i}.$$

Note that $S_n \to \infty$ as $n \to \infty$.

Let $h\colon [0,\infty) \to [0,1)$ be the 'fractional part' function:

$$h(x) = \{x\} = x - [x].$$

Define functions $f_n\colon [0,\infty) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [h(S_{n-1}), h(S_n)] \\ 0 & \text{otherwise} \end{cases}$$

with understanding that if $h(S_{n-1}) > h(S_n)$, then

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [h(S_{n-1}), 1] \cup [0, h(S_n)] \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\int_{[0,1]} f_n \, d\mu = \frac{1}{n} \to 0.$$

At the same time, since $S_n \to \infty$ as $n \to \infty$, for every $x \in [0,1)$ there exists infinitely many n's such that $f_n(x) = 1$ (as well as infinitely many m's such that $f_m(x) = 0$).

The point $x = 1$ is not covered by the above analysis, so we can simply set $f_n(1) = 0$ for odd n's and $f_n(1) = 1$ for even n's (this will not affect the values of the integrals).