MA 645-2F (Real Analysis), Dr. Chernov — Midterm test

Full credit for 5 problems. — Thu, Oct 4, 2012

1. (a) Let $\mathfrak{M}_1$ denote the Lebesgue σ-algebra in $\mathbb{R}$ and $\mathfrak{M}_2$ the Lebesgue σ-algebra in $\mathbb{R}^2$. Suppose $A \subset \mathbb{R}$ is a **non-measurable** set, i.e., $A \notin \mathfrak{M}_1$. Is the set

$$B = \{(x, y) \colon x \in A,\ y = 1\}$$

measurable in $\mathbb{R}^2$, i.e., is it true that $B \in \mathfrak{M}_2$?

(b) Describe all the sets $A \subset \mathbb{R}$ such that the set

$$B = \{(x, y) \colon x \in A,\ y = 1\}$$

is measurable in $\mathbb{R}^2$, i.e., $B \in \mathfrak{M}_2$.

Solution: The set B is a subset of the line $L = \{y = 1\}$. Every line has Lebesgue measure zero in $\mathbb{R}^2$. This can be stated without proof, as you learned in Calculus-III that every line (as well as every curve) has no area (its area is zero), and the Lebesgue measure generalizes the concept of area.

Alternatively, you can prove that $\mathbf{m}(L) = 0$, where $\mathbf{m}$ denotes the Lebesgue measure in $\mathbb{R}^2$, as follows. Every line L is a countable union of line segments, $L = \cup_{n=1}^{\infty} I_n$, where I_n is a segment of the line L (i.e., the length of I_n is finite). Then $\mathbf{m}(I_n) = 0$, as was shown in one of the homework exercises. Now by the σ-subadditivity $\mathbf{m}(L) \leq \sum \mathbf{m}(I_n) = \sum 0 = 0$.

The Lebesgue measure $\mathbf{m}$ in $\mathbb{R}^2$ is complete, so every subset of a set of measure zero is measurable. Thus $B \in \mathfrak{M}_2$.

Part (b): it is now clear that any subset $A \subset \mathbb{R}$ has the claimed property.

2. (a) Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose A_n, $n \geq 1$, are full measure sets. Prove that $A = \cap_{n=1}^{\infty} A_n$ is a full measure set.

(b) Let $(X, \mathfrak{M}, \mu)$ be a measure space. Denote by $\mathfrak{M}_0$ the collection of all null sets and all full measure sets in X. Prove that $\mathfrak{M}_0$ is a σ-algebra.

Solution:

Note that by definition A is a full measure set iff A^c is a null set, i.e., $\mu(A^c) = 0$. It is not correct to say that A is a full measure set iff $\mu(A) = \mu(X)$. This would be only true if $\mu(X) < \infty$. If $\mu(X) = \infty$, then not every set $A \subset X$ with $\mu(A) = \infty$ is a full measure set. Example: $X = \mathbb{R}$, μ is the Lebesgue measure, and A is a set of positive real numbers.

(a) We have $\mu(A_n^c) = 0$ for every $n \geq 0$. Also recall that $A^c = \left(\cap_{n=1}^{\infty} A_n\right)^c = \cup_{n=1}^{\infty} A_n^c$. Now by the σ-subadditivity

$$\mu\left(\cup_{n=1}^{\infty} A_n^c\right) \leq \sum_{n=1}^{\infty} \mu(A_n^c) = \sum_{n=1}^{n} 0 = 0,$$

hence $\mu(A^c) = 0$. Therefore A is a full measure set.

(b) [This was graded as a bonus question.] A σ-algebra must contain X, $\emptyset$ and be closed under countable unions and complements. Obviously, $\emptyset$ is a null set and X is a full measure set. A complement of a null set is a full measure set and vice versa. Now consider a countable union $B = \cup B_n$ of sets B_n that are either null sets or full measure sets. If at least one B_n is a full measure set, then note that their union B will contain B_n (i.e., $B \supset B_n$), hence B will also be a full measure set. Lastly, if all B_n's are null sets, then as we have shown in part (a), their union will be a null set.

3. Let $(X, \mathfrak{M})$ be a measurable space and $f_n\colon X \to [-\infty, \infty]$ are measurable functions for $n \geq 1$. Prove that the set

$$E = \{x \in X\colon \lim_{n\to\infty} f_n(x) \text{ exists and is finite}\}$$

is measurable, i.e., $E \in \mathfrak{M}$.

Solution:
Let

$$g(x) = \limsup_{n\to\infty} f_n(x)$$

and

$$h(x) = \liminf_{n\to\infty} f_n(x).$$

We know that both functions $g(x)$ and $h(x)$ are measurable. Also recall that $\lim_{n\to\infty} f_n(x)$ exists if and only if $g(x) = h(x)$.

Now the set

$$E' = \{x \in X\colon g(x) = h(x)\}$$

is measurable, as it was proved in a homework exercise. Lastly,

$$E = E' \setminus \{x \in X\colon g(x) = \pm\infty\}.$$

The last set is obviously measurable, so E is measurable, too.

4. Let $\mathbf{m}$ denote the Lebesgue measure on $\mathbb{R}$. Suppose $f\colon \mathbb{R} \to [0, \infty)$ is a continuous function such that $f \in L^1_{\mathbf{m}}(\mathbb{R})$. Is it true that $\lim_{x\to\infty} \frac{f(x)}{x^{10}} = 0$? (Prove or give a counterexample.)

Solution:

A typical mistake is to argue as follows: assume (by way of contradiction) that $\lim_{x\to\infty} \frac{f(x)}{x^{10}} \neq 0$... This of quickly leads to a contradiction. But this DOES NOT prove that $\lim_{x\to\infty} \frac{f(x)}{x^{10}} = 0$. Why not? Because the limit may not even exist!

A counterexample is any function $f(x) \geq 0$ that is very close to zero (or is zero) most of the time but occasionally surges high up and quickly drops down to zero (its graph has very narrow and very high "bumps"). A rough sketch would be enough to get good partial credit.

A formal example can be presented as follows:

$$f(x) = \sum_{n=1}^{\infty} n^{11} \chi_{[n, n+n^{-13}]}$$

The factor n^{11} ensures that the "bumps" are high enough for the $\lim_{x\to\infty} \frac{f(x)}{x^{10}}$ to fail to exist (in fact, the corresponding $\limsup$ would be infinite). The widths of the bumps, n^{-13}, are selected so that the area under the nth bump is n^{-2}. Since the series $\sum_{n=1}^{\infty} n^{-2}$ converges, we have $f \in L^1_{\mathbf{m}}(\mathbb{R})$.

In the above example $f(x)$ is a discontinuous function, but it is good enough for you to get full credit... To make $f(x)$ continuous, you can just replace rectangular bumps with triangular bumps or trapezoidal bumps, etc.

5. Let $f\colon \mathbb{R} \to [0,\infty]$ be a measurable and <u>bounded</u> function. Is it always true that

$$\int_{\mathbb{R}} f\,d\mathbf{m} = \inf \int_{\mathbb{R}} s\,d\mathbf{m}$$

where the infimum is taken over all simple measurable functions $s\colon \mathbb{R} \to [0,\infty]$ such that $f \le s$? Prove or give a counterexample. Justify your conclusions.
(Here again $\mathbf{m}$ denotes the Lebesgue measure on $\mathbb{R}$.)

Solution:

This would be true if the measure of the whole space was finite. But here it is infinite: $\mathbf{m}(\mathbb{R}) = \infty$, so the claim is not always true.

A counterexample is any bounded function $f(x) \in L^1_{\mathbf{m}}(\mathbb{R})$ such that $f(x) > 0$ for all $x \in \mathbb{R}$. For example, $f(x) = e^{-x^2}$ or $f(x) = \frac{1}{1+x^2}$, which you might remember from Calculus or Probability Theory. If you do not remember any, you can construct f as follows:

$$f = \sum_{n=-\infty}^{\infty} 2^{-|n|}\,\chi_{[n,n+1)}$$

For this function we have

$$\int_{\mathbb{R}} f\,d\mathbf{m} = \sum_{n=-\infty}^{\infty} 2^{-|n|} = 3 < \infty$$

Now since $f(x) > 0$ for all $x \in \mathbb{R}$, then any simple function $s \ge f$ must also be positive *everywhere*, so that in the representation

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

all the values α_i are positive: $\alpha_i > 0$ for all $i = 1, \ldots, n$. At the same time at least one subset A_{i_0} must have infinite measure, i.e., $\mathbf{m}(A_{i_0}) = \infty$. Therefore

$$\int s\,d\mathbf{m} = \sum_{i=1}^{n} \alpha_i \mathbf{m}(A_i) \ge \alpha_{i_0}\mathbf{m}(A_{i_0}) = \infty$$

for every simple function $s \ge f$. Thus we have $\inf \int_{\mathbb{R}} s\,d\mathbf{m} = \infty$, while $\int_{\mathbb{R}} f\,d\mathbf{m} < \infty$.

6. Let $\mathbf{m}$ denote the Lebesgue measure on $\mathbb{R}$ and $E = [-1, 1]$. Find the limit

$$\lim_{n\to\infty} \int_E \cos x^n \, d\mathbf{m}.$$

Provide a rigorous proof with careful justification at all steps.

Solution:
For all $x \in (-1, 1)$ we have

$$\lim_{n\to\infty} x^n = 0.$$

Since cos is a continuous function, we have

$$\lim_{n\to\infty} \cos x^n = \cos 0 = 1.$$

At the point $x = 1$ we have $\lim_{n\to\infty} x^n = 1$ and at the point $x = -1$ the limit of x^n does not exist. But the two-point set $\{1, -1\}$ has measure zero, so it does not affect the values of our integrals and can be ignored (the "almost everywhere" principle). Thus we can work on the open interval $E' = (-1, 1)$.

Note that $|\cos x^n| \le g(x) = 1$ and

$$\int_{E'} |g(x)| \, d\mathbf{m} = 1 \times |E'| = 2$$

therefore $g \in L^1_{\mathbf{m}}(E')$. By the Lebesgue Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_{E'} \cos x^n \, d\mathbf{m} = \int_{E'} \lim_{n\to\infty} \cos x^n \, d\mathbf{m} = \int_{E'} 1 \, d\mathbf{m} = 2.$$