MA 645-4A (Real Analysis), Dr. Chernov — Final exam

Full credit for 6 (out of 8) problems. — Thu, Dec 14, 2006

1. Prove that

$$\lim_{n\to\infty}\int_0^1 \frac{n^{3/4}x}{1+nx^2}\,dx = 0.$$

2. Prove or disprove the following statement:
If $(X, \mathfrak{M}, \mu)$ is a measure space, φ is a convex function on (a, b) and $f\colon X \to (a, b)$ is an integrable function, i.e. $f \in L^1_\mu(X)$, then

$$\varphi\Big(\int_X f \, d\mu\Big) \leq \int_X (\varphi \circ f) \, d\mu.$$

3. Given that $\int_0^\infty e^{-x} \sin x \, dx = \frac{1}{2}$, prove that

$$\int_0^\infty e^{-x} \sqrt{3 + 2 \sin x} \, dx \leq 2.$$

4. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of continuous functions on $[0, \infty)$ that uniformly converges to $f(x)$ as $n \to \infty$. Assume that f_n $(n \geq 1)$ and f are Lebesgue integrable on $[0, \infty)$. Is it always true that

$$\lim_{n \to \infty} \int_0^{\infty} |f_n - f| \, dm = 0.$$

Prove or give a counterexample.

5. Let $f \in L^1_\mu(X) \cap L^\infty_\mu(X)$. Prove that

$$\|f\|_p \leq \|f\|_\infty^{1/q} \, \|f\|_1^{1/p}.$$

for any pair of conjugate exponents $p, q > 1$ (i.e. such that $\frac{1}{p} + \frac{1}{q} = 1$). Hint: consider the function $g = f/\|f\|_\infty$.

6. Suppose $f \in L^2(\mathbb{R})$. Prove that

$$\lim_{n\to\infty} \int_n^{n+1} f\, dm = 0.$$

7. Let $p > 1$ and $f \in L^p\big([-1,1]\big)$, i.e.

$$\int_{[-1,1]} |f|^p \, dm < \infty.$$

Let $I_n = (-\frac{1}{n}, \frac{1}{n})$ and $\gamma = \frac{p-1}{p}$. Show that

$$\lim_{n\to\infty} n^\gamma \int_{I_n} |f| \, dm = 0.$$

8. Let $a_1, \ldots, a_n$ be positive numbers. Prove that their harmonic mean does not exceed their arithmetic mean, i.e.

$$\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}\right)^{-1} \leq \frac{1}{n}\sum_{i=1}^{n} a_i$$