1. Is the following true or false? (Prove or give a counterexample.)

(a) If $f(x)$ is continuous a.e. on $[0,1]$, then there exists a continuous function $g(x)$ on $[0,1]$ such that $f = g$ a.e. on $[0,1]$.

(b) If $f(x)$ is continuous everywhere on $[0,1]$ and $f(x) = g(x)$ a.e. on $[0,1]$, then $g(x)$ is continuous a.e. on $[0,1]$.

(c) If μ is a measure on $\mathbb{R}$ (not necessarily Lebesgue measure), then there exists a countable partition $\mathbb{R} = \uplus_{n=1}^{\infty} A_n$ such that $\mu(A_n) < \infty$ for all n.

Solutions:

(a) False. For example, let $f = \chi_{[0,\frac{1}{2}]}$. If $g = f$ a.e., then the sets of points $\{x \in [0,1] \colon g(x) = f(x)\}$ is dense. Thus there are sequences $a_n \to \frac{1}{2}$, $a_n < \frac{1}{2}$, and $b_n \to \frac{1}{2}$, $b_n > \frac{1}{2}$, such that $g(a_n) = f(a_n) = 1$ and $g(b_n) = f(b_n) = 0$ for all $n \geq 1$. Thus the one-sided limits of $g(x)$ at the point $x = \frac{1}{2}$ must be different (the left-sided limit must be $= 1$ and the right-sided limit must be $= 0$), so g cannot be continuous.

(b) False. For example, let $f \equiv 0$ on $[0,1]$ and $g = \chi_{\mathbb{Q}\cup[0,1]}$ be the Dirichlet function restricted to $[0,1]$. Then $f(x) = g(x)$ for all irrational $x \in [0,1]$, thus a.e., but $g(x)$ is discontinuous at every point $x \in [0,1]$. (At every $a \in [0,1]$ we have $\limsup_{x\to a} f(x) = 1$ and $\liminf_{x\to a} f(x) = 0$.)

(c) False. For example, let μ be the counting measure on $\mathbb{R}$. Then $\mu(A_n) < \infty$ if and only if A_n is a finite set. If $\mathbb{R}$ was a countable union of finite sets, it would be a countable set, but we know that $\mathbb{R}$ is uncountable.

2. Is the modified Dirichlet function lower semicontinuous? Is it upper semicontinuous? Justify your answers. (Modified Dirichlet function is described in Chapter 10 of the extended class notes.)

Solution (we denote this function by f):

It is not lower semicontinuous. Indeed, for each $a \in (0,1)$ the set $f^{-1}(a,\infty)$ is finite, hence not open in $[0,1]$.

It is upper semicontinuous. Indeed, the set $f^{-1}(-\infty, a)$ is always open: for $a \leq 0$ it is empty, for $a \in (0,1]$ this set is $[0,1]$ without a finite number of points, and for $a > 1$ this set is $[0,1]$.

3. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $A_1, A_2, \ldots$ measurable sets. Suppose

$$\mu(A_i \cap A_j) = 0$$

for all $i \neq j$. Is it always true that

$$\mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)?$$

Prove or give a counterexample.

Solution: Define $A_i' = A_i \setminus \cup_{j=1}^{i-1} A_j$, then A_i' are disjoint sets and

$$\cup_{n=1}^{\infty} A_n = \uplus_{i=1}^{\infty} A_i'$$

is a disjoint union of sets, thus

$$\mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n').$$

It remains to show that $\mu(A_i) = \mu(A_i')$ for every $i \geq 1$.

We have

$$A_i = A_i' \cup (A_i \cap A_1) \cup \cdots \cup (A_i \cap A_{i-1})$$

thus by the subadditivity of measures

$$\mu(A_i) \leq \mu(A_i') + \sum_{j=1}^{i-1} \mu(A_i \cap A_j) = \mu(A_i')$$

because $\mu(A_i \cap A_j) = 0$. On the other hand, $A_i' \subset A_i$, hence $\mu(A_i') \leq \mu(A_i)$.

4. Suppose $f(x)$ is Riemann integrable on $[0,1]$ and $f(r) = 0$ for every rational number $r \in [0,1]$. Find the Lebesgue integral $\int_{[0,1]} f \, d\mathbf{m}$.

Solution: Since f is Riemann integrable, it is a.e. continuous, i.e., the set

$$E_f = \{x \in [a,b] \colon f \text{ is continuous at } x\}$$

has full measure. For every $x \in E_f$ there exists a sequence of rational numbers $r_n \in [0,1]$ such that $r_n \to x$. Then by the continuity of f at x we have $f(x) = \lim_{n\to\infty} f(r_n) = 0$. Thus $f = 0$ almost everywhere on $[0,1]$. Therefore the Lebesgue integral of f is zero. Since f is Riemann integrable, its Riemann integral is equal to its Lebesgue integral, hence it is zero, too.

5. Let $f_1, f_2, \ldots$ be measurable functions on $\mathbb{R}$. Suppose $f_n \to f$ in measure and $f_n \to g$ in measure.

(a) Prove that f and g are measurable functions.

(b) Is it always true that $f(x) = g(x)$ a.e. on $\mathbb{R}$? (Prove or give a counterexample.)

Solution:

Since f_n converges to f in measure, there exists a subsequence $\{f_{n_i}\}$ that converges to f a.e. Let A denote the set of points where $\lim_{i\to\infty} f_{n_i}(x) = f(x)$. Since the pointwise limit of measurable functions is a measurable function, the restriction of f to A is measurable. Finally, since the complement A^c is a null set, the whole function f is measurable.

The same argument applies to g.

Now let again $\{f_{n_i}\}$ be subsequence $\{f_{n_i}\}$ that converges to f a.e. (more precisely, on a full measure set A). Obviously, the subsequence $\{f_{n_i}\}$ converges to g in measure, just as the whole sequence $\{f_n\}$ does. Thus there is a subsubsequence $\{f_{n_{i_j}}\}$ of the subsequence $\{f_{n_i}\}$ that converges to g almost everywhere. Let B denote the set of points where $\lim_{j\to\infty} f_{n_{i_j}}(x) = g(x)$. Then for every point $x \in A \cap B$ the subsubsequence $f_{n_{i_j}}(x)$ converges to both $f(x)$ and $g(x)$. Therefore $f(x) = g(x)$ on the intersection $A \cap B$, which is a set of full measure because so are A and B.

6. Suppose $f\colon [0,1] \to [0,\infty]$ is a measurable function. Prove that

$$\left[\int_{[0,1]} f(x)\cos x\, d\mathbf{m}\right]^2 + \left[\int_{[0,1]} f(x)\sin x\, d\mathbf{m}\right]^2 \le \left[\int_{[0,1]} f(x)\, d\mathbf{m}\right]^2$$

Solution: By the Schwarz inequality

$$\left[\int_{[0,1]} \sqrt{f(x)}\sqrt{f(x)}\cos x\, d\mathbf{m}\right]^2 \le \int_{[0,1]} f(x)\, d\mathbf{m} \cdot \int_{[0,1]} f(x)\cos^2 x\, d\mathbf{m}$$

and similarly

$$\left[\int_{[0,1]} \sqrt{f(x)}\sqrt{f(x)}\sin x\, d\mathbf{m}\right]^2 \le \int_{[0,1]} f(x)\, d\mathbf{m} \cdot \int_{[0,1]} f(x)\sin^2 x\, d\mathbf{m}$$

Adding these up gives the desired inequality.

Note that it may be tempting to use the Schwarz inequality as follows:

$$\left[\int_{[0,1]} f(x)\cos x\, d\mathbf{m}\right]^2 \le \int_{[0,1]} f^2(x)\, d\mathbf{m} \cdot \int_{[0,1]} \cos^2 x\, d\mathbf{m}$$

and similarly

$$\left[\int_{[0,1]} f(x)\sin x\, d\mathbf{m}\right]^2 \le \int_{[0,1]} f^2(x)\, d\mathbf{m} \cdot \int_{[0,1]} \sin^2 x\, d\mathbf{m}.$$

Adding these up gives

$$\left[\int_{[0,1]} f(x)\cos x\, d\mathbf{m}\right]^2 + \left[\int_{[0,1]} f(x)\sin x\, d\mathbf{m}\right]^2 \le \int_{[0,1]} f^2(x)\, d\mathbf{m}$$

but this is **weaker** than the desired inequality.

7. Assume that f and f_n are measurable functions on $[0,1]$ and that $f_n \geq 0$ a.e. on $[0,1]$. Suppose $f_n(x) \to f(x)$ a.e. on $[0,1]$.

(a) Does the following convergence always take place? (Prove or give a counterexample.)

$$\int_{[0,1]} f_n \, d\mathbf{m} \to \int_{[0,1]} f \, d\mathbf{m}.$$

(b) Does the following convergence always take place? (Prove or give a counterexample.)

$$\int_{[0,1]} f_n e^{-f_n} \, d\mathbf{m} \to \int_{[0,1]} f e^{-f} \, d\mathbf{m}.$$

Note: question (a) is very easy and given only as "introduction" to question (b).

Solutions:

(a) is false. A standard example: $f_n = n\chi_{(0,\frac{1}{n})}$ and $f \equiv 0$. Then $f_n(x) \to f(x)$ for every $x \in [0,1]$, but

$$\int_{[0,1]} f_n \, d\mathbf{m} = 1 \neq 0 = \int_{[0,1]} f \, d\mathbf{m}.$$

(b) is true. Indeed,

$$f_n(x)e^{-f_n(x)} \to f(x)e^{-f(x)} \qquad \text{a.e.}$$

(quite obviously), and the integrands here are bounded by one integrable function:

$$0 \leq f_n(x)e^{-f_n(x)} \leq g(x) = 1.$$

In fact, the bound can be tightened: the function te^{-t} for $t \geq 0$ reaches its maximum at $t = 1$ and its maximum value is $e^{-1} < 1$.

Now the Lebesgue Dominated Convergence applies and completes the solution.

8. Consider the following functions on $[0,1]$:

$$f_n(x) = \frac{1+n^2x^2}{(1+x^2)^n}$$

(a) Find the pointwise limit of $f_n(x)$ for all $x \in [0,1]$.

(b) Find the limit of integrals:

$$\lim_{n\to\infty} \int_{[0,1]} f_n \, d\mathbf{m}$$

Note: question (a) is easy and given only as "introduction" to question (b).

Solutions:

(a) The pointwise limit of $f_n(x)$ is the function $f(x)$ such that $f(0) = 1$ and $f(x) = 0$ for all $x > 0$. This is a simple calculus exercise.

(b) We claim that

$$\lim_{n\to\infty} \int_{[0,1]} f_n \, d\mathbf{m} = \infty.$$

Indeed, a quick inspection shows that the integrand takes high values for $x \sim 1/\sqrt{n}$. So let us estimate the integral from below by

$$\int_{[0,1]} \frac{1+(nx)^2}{(1+x^2)^n} \, d\mathbf{m} \geq \int_{[0,\frac{1}{\sqrt{n}}]} \frac{(nx)^2}{(1+x^2)^n} \, d\mathbf{m}$$

Now for all $x \in [0, \frac{1}{\sqrt{n}}]$ we have

$$\frac{(nx)^2}{(1+x^2)^n} \geq \frac{(nx)^2}{(1+1/n)^n} \geq \frac{(nx)^2}{e}$$

because $(1+1/n)^n \leq e$. Therefore

$$\int_{[0,1]} \frac{1+(nx)^2}{(1+x^2)^n} \, d\mathbf{m} \geq \frac{1}{e} \int_{[0,\frac{1}{\sqrt{n}}]} (nx)^2 \, d\mathbf{m} = \frac{\sqrt{n}}{3e} \to \infty.$$