1 (Due 9/5). Prove that every countable set $A \subset X$ is measurable and $\mu(A) = 0$.

2 (Bonus). Let $A \subset X$ consist of points (x, y) such that either x or y is a rational number. Is A measurable? What is its Lebesgue measure?

3 (Due 9/5). Prove that every open set $A \subset X$ is measurable (hint: represent it by a countable union of rectangles). Prove that every closed set $A \subset X$ is measurable.

4 (Due 9/14). Let $\mathfrak{L}$ denote the set of all Lebesgue measurable sets (in $\mathbb{R}$). Prove that $\text{card}(\mathfrak{L}) > \mathfrak{C}$.

5 (Due 9/14). Let $\mathfrak{U}$ denote the set of all open subsets $U \subset \mathbb{R}$. Prove that $\text{card}(\mathfrak{U}) = \mathfrak{C}$. Do the same for open sets in $\mathbb{R}^2$.

6 (Due 9/14). Let $X = \{1, 2, 3\}$. Construct all σ-algebras of X.

7 (Due 9/14). Let $X = [0, 1]$ and $\mathfrak{G}$ consist of all one-point sets, i.e. $\mathfrak{G} = \big\{\{x\}, x \in X\big\}$. Describe the σ-algebra $\mathfrak{M}(\mathfrak{G})$.

8 (Due 9/14). Show that every Borel set in $\mathbb{R}$ is Lebesgue measurable, but not vice versa.

9 (Bonus). Does there exist an infinite σ-algebra which has only countably many members?

10 (Due 9/14). Show that the assumption $\mu(A_1) < \infty$ in Theorem 2.10 is indispensable. Hint: consider the counting measure on $\mathbb{N}$ and take sets $A_n = \{n, n+1, \dots\}$.

11 (Due 9/14). Let $X \subset \mathbb{R}^2$ be a rectangle. Verify that the collection of all subrectangles $R \subset X$ is a semi-algebra.

12 (Due 9/21). Prove that $A \in \mathfrak{M}$ if and only if χ_A is measurable.

13 (Due 9/21). Let $(X, \mathfrak{M})$ be a measurable space and $f \colon X \to [-\infty, \infty]$. Prove that f is measurable iff $f^{-1}\big([-\infty, x]\big)$ is a measurable set for every $x \in \mathbb{R}$.

14 (Due 9/21). Let $(X, \mathfrak{M})$ be a measurable space and $f\colon X \to [-\infty, \infty]$. Prove that f is measurable iff $f^{-1}\big([-\infty, x]\big)$ is a measurable set for every rational $x \in \mathbb{Q}$.

15 (Bonus). Let $(X, \mathfrak{M})$ be a measurable space and $f\colon X \to [-\infty, \infty]$ and $g\colon X \to [-\infty, \infty]$ two measurable functions. Prove that the sets

$$\{x\colon f(x) < g(x)\} \qquad \text{and} \qquad \{x\colon f(x) = g(x)\}$$

are measurable.

16 (Due 9/26). Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is a Borel function.

17 (Due 9/26). Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a Borel function.

18 (Bonus). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function, i.e. $f(x_1) \leq f(x_2)$ for $x_1 \leq x_2$. Show that f is a Borel function.

19 (Due 10/3). Let $x_0 \in X$ and $\mu = \delta_{x_0}$ the δ-measure. Assume that $\{x_0\} \in \mathfrak{M}$. Show that for every measurable function $f\colon X \to [0, \infty]$ we have

$$\int_X f\, d\mu = f(x_0).$$

20 (Due 10/3). Let $X = \mathbb{N}$ and μ the counting measure on the σ-algebra $\mathfrak{M} = 2^{\mathbb{N}}$. Show that for every function $f\colon X \to [0, \infty]$ we have

$$\int_X f\, d\mu = \sum_{n=1}^{\infty} f(n).$$

21 (Due 10/3). Let $f\colon X \to [0, \infty]$ and $\int_X f\, d\mu = 0$. Show that $\mu\{x\colon f(x) \neq 0\} = 0$.

22 (Due 10/3). Let $E \subset X$ be such that $\mu(E) > 0$ and $\mu(E^c) > 0$. Put $f_n = \chi_E$ if n is odd and $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

23 (Due 10/3). Construct an example of a sequence of nonnegative measurable functions $f_n \colon X \to [0, \infty)$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists pointwise, but

$$\int_X f \, d\mu < \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

24 (Due 10/5). Let $f, g \in L^1(\mu)$ be real-valued functions and $f \leq g$. Show that

$$\int_X f \, d\mu \leq \int_X g \, d\mu.$$

25 (Due 10/5). Let $f_n \colon X \to [0, \infty]$ be a sequence of measurable functions such that $f_1 \geq f_2 \geq \cdots \geq 0$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. Suppose $f_1 \in L^1(\mu)$. Show that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

26 (Due 10/10). Let f be a function as above. Fix a $y \in Y$ and define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \setminus N \\ y & \text{if } x \in N \end{cases}.$$

Show that $\tilde{f} \colon X \to Y$ is measurable.

27 (Due 10/10). Let f be a function as above and μ a complete measure. Show that

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \setminus N \\ \text{any point of } Y & \text{if } x \in N \end{cases}.$$

Show that $\tilde{f} \colon X \to Y$ is measurable.

28 (Due 10/10). Suppose $\mu(X) < \infty$. Let a sequence $\{f_n\}$ of bounded complex measurable functions uniformly converge to f on X. Prove that $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$. Show that the assumption $\mu(X) < \infty$ cannot be omitted.

29 (Due 10/10). Give an example of a sequence of complex measurable functions $f_n : X \to \mathbb{C}$ (i.e., defined on the entire X) such that

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty$$

but the series $f(x) = \sum_{n=1}^{\infty} f_n$ diverges for some $x \in X$.

30 (Due 10/12). Suppose $\mu(X) < \infty$ and f_n are measurable functions defined a.e. on X. Prove that if $f_n \to f$ a.e. on X, then $f_n \to f$ in measure. What happens if $\mu(X) = \infty$?

31 (Due 10/12). In Theorem 4.26, let A be the set of points which belong to infinitely many of the sets E_k. Show that

$$A = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k.$$

Use this fact to prove the theorem without any reference to integration. Hint: recall that every measure μ is countably subbaditive, i.e. for any $E_n \in \mathfrak{M}$ (not necessarily disjoint) $\mu(\cup E_n) \leq \sum \mu(E_n)$.

32 (Bonus). Prove that if $f_n \to f$ in measure, then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ a.e. on X. Hint: use Theorem 4.26.

33 (Bonus). Suppose $f \in L^1(\mu)$. Prove that $\forall \varepsilon > 0 \ \exists \delta > 0$ such that $\int_E |f| \, d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

34 (Due 10/19). Show that the counting measure on $\mathbb{R}^k$ is not regular.

35 (Due 10/19). Find examples of Lebesgue measurable sets $E_1, E_2 \subset \mathbb{R}$ such that

$$\mu(E_1) < \inf\{\mu(A) \colon E \subset A, \quad A \text{ closed}\}$$
$$\mu(E_2) > \sup\{\mu(V) \colon V \subset E, \quad V \text{ open}\}.$$

36 (Due 10/19). Extend this theorem to $\mathbb{R}^2$: show that if μ^* is a translation invariant Borel measure on $\mathbb{R}^2$ such that $\mu(R) < \infty$ for at least one open rectangle $R \neq \emptyset$, then there exists a constant c such that $\mu^*(E) = c\mu(E)$ for every Borel set E.

37 (Bonus). Let $f \colon \mathbb{R} \to [0, \infty]$ be a Borel function, $f \in L^1(\mu)$, and consider the measure $\rho(E) = \int_E f \, d\mu$, where μ is the Lebesgue measure. Prove that ρ is regular. Hint: use the result of Exercise 33.

38 (Due 11/2). Let $s\colon [a,b] \to \mathbb{R}$ be a simple Lebesgue measurable function. Show that for every $\varepsilon > 0$ there is a step function $\varphi\colon [a,b] \to \mathbb{R}$ and a Lebesgue measurable set $E \subset [a,b]$ such that $s(x) = \varphi(x)$ on E and $\mu\big([a,b] \setminus E\big) < \varepsilon$. Hint: use the regularity of μ.

39 (Due 11/2). Let $f\colon [a,b] \to \mathbb{R}$ be a Lebesgue measurable function. Show that for every $\varepsilon > 0$ there is a step function $g\colon [a,b] \to \mathbb{R}$ such that

$$\mu\big\{x \in [a,b] \colon |f(x) - g(x)| \geq \varepsilon\big\} < \varepsilon.$$

Hint: first show that $|f| \leq M$ except for a set of small measure, then use the previous exercise.

40 (Bonus). Let $f \in L^1$ with respect to the Lebesgue measure μ on $\mathbb{R}$. Prove that there is a sequence $\{g_n\}$ of step functions such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f - g_n| \, d\mu = 0.$$

Hint: use the previous exercise.

41 (Due 11/2). Prove that if $f\colon X \to \mathbb{R}$ is upper (lower) semicontinuous and X is compact, then f is bounded above (below) and attains its maximum (minimum).

42 (Due 11/9). Find a function $f(x)$ on $[0,\infty)$ such that the improper Riemannian integral $\int_0^\infty f(x)\,dx = \lim_{A\to\infty} \int_0^A f(x)\,dx$ exists (is finite), but f is not Lebesgue integrable.

43 (Due 11/14). Prove that the supremum of any collection of convex functions on (a,b) is convex on (a,b) (assume that the supremum is finite). Prove that a pointwise limit of a sequence of convex functions is convex.

44 (Due 11/14). Let φ be convex on (a,b) and ψ convex and nondecreasing on the range of φ. Prove that $\psi \circ \varphi$ is convex on (a,b). For $\varphi > 0$, show that the convexity of $\log \varphi$ implies the convexity of φ, but not vice versa.

45 (Bonus). Assume that φ is a continuous real function on (a,b) such that

$$\varphi\Big(\frac{x+y}{2}\Big) \leq \frac{1}{2}\,\varphi(x) + \frac{1}{2}\,\varphi(y)$$

for all $x, y \in (a,b)$. Prove that φ is convex. (The conclusion does not follow if continuity is omitted from the hypotheses.)

46 (Due 11/21). Prove that equality in the Minkowski inequality holds if and only if there exist $\alpha, \beta \geq 0$, not both equal to 0, such that $\alpha f = \beta g$ a.e.

47 (Due 11/21). Suppose $\mu(\Omega) = 1$ and suppose f and g are two positive measurable functions on Ω such that $fg \geq 1$. Prove that

$$\int_\Omega f \, d\mu \cdot \int_\Omega g \, d\mu \geq 1.$$

48 (Due 11/21). Suppose $\mu(\Omega) = 1$ and $h \colon \Omega \to [0, \infty]$ is measurable. Denote $A = \int_\Omega h \, d\mu$. Prove that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2} \, d\mu \leq 1 + A.$$

49 (Bonus). If μ is Lebesgue measure on $[0, 1]$ and if h is a continuous function on $[0, 1]$ such that $h = f'$, then the inequalities in the previous exercise have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

50 (Due 11/28). When does one get equality in $\|fg\|_1 \leq \|f\|_\infty \|g\|_1$?

51 (Due 11/28). Suppose $f \colon X \to \mathbb{C}$ is measurable and $\|f\|_\infty > 0$. Define

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \qquad (0 < p < \infty)$$

and consider the set $E = \{p \colon \varphi(p) < \infty\}$. Each of the following questions is graded as a separate exercise. Question (c) and (e) are **bonus** problems.

(a) Let $r < p < s$ and $r, s \in E$. Prove that $p \in E$.

(b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \leq \max\big(\|f\|_r, \|f\|_s\big)$. Show that this implies the inclusion $L^r_\mu \cap L^s_\mu \subset L^p_\mu$.

(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

52 (Due 12/5). Let $\mu(X) = 1$. Each of the following questions is graded as a separate exercise.

(a) Prove that $\|f\|_r \leq \|f\|_s$ if $0 < r < s \leq \infty$.

(b) Under what condition does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(c) Prove that $L^r_\mu \supset L^s_\mu$ if $0 < r < s$. If $X = [0, 1]$ and μ is the Lebesgue measure, show that $L^r_\mu \neq L^s_\mu$.

53 (Bonus). For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ if $r \neq s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

54 (Due 12/5). (a) Show that $\int_0^{\frac{\pi}{2}} \sqrt{x \sin x}\, dx < \frac{\pi}{2\sqrt{2}}$;
(b) Show that $\left[\int_0^1 x^{\frac{1}{2}}(1-x)^{-\frac{1}{3}}\, dx\right]^3 \leq \frac{8}{5}$.

55 (Bonus). Suppose $1 < p < \infty$ and $f \in L^p((0, \infty))$ relative to the Lebesgue measure. Define

$$F(x) = \frac{1}{x}\int_0^x f(t)\, dt \qquad (0 < x < \infty).$$

Prove Hardy's inequality

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p$$

which shows that the mapping $f \to F$ carries L^p into L^p. [Hint: assume first that $f \geq 0$ and $f \in C_c((0, \infty))$, i.e. the support of f is a finite closed interval $[a, b] \subset (0, \infty)$. Then use integration by parts:

$$\int_\varepsilon^A F^p(x)\, dx = -p\int_\varepsilon^A F^{p-1} x F'(x)\, dx$$

where $\varepsilon < a$ and $A > b$. Note that $xF' = f - F$, and apply Hölder inequality to $\int F^{p-1} f\, dx$.]

56 (Due 1/16). Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1_\mu$ define

$$\mu_f(E) = \int_E f\, d\mu.$$

Show:

(a) μ_f is a complex measure;
(b) $|\mu_f| = \mu_{|f|}$, assuming that f is real-valued;
(c) $|\mu_f| = \mu_{|f|}$, now for a general $f \in L^1(\mu)$.

57 (Bonus). Prove that the space $\mathbb{M}(X, \mathfrak{M})$ with the norm $\|\mu\|$ is a Banach space, i.e. it is a complete metric space (every Cauchy sequence converges to a limit). Hint: given a Cauchy sequence of complex measures $\{\mu_n\}$ you need to construct the limit measure μ and prove that $\|\mu_n - \mu\| \to 0$ as $n \to \infty$.

58 (Due 1/25). Let μ be a positive measure on $(X, \mathfrak{M})$ and $f, g \in L^1_\mu$. Prove that

(a) μ_f is concentrated on A if and only if $\mu\{x \in A^c \colon f(x) \neq 0\} = 0$;

(b) $\mu_f \perp \mu_g$ if and only if $\mu\{x \in X \colon f(x)g(x) \neq 0\} = 0$;

(c) if $f \geq 0$, then

$$\mu \ll \mu_f \iff f(x) > 0 \text{ for } \mu - \text{a.e. } x \in X.$$

59 (Due 1/25). Let λ be a positive measure on $(X, \mathfrak{M})$. Prove that λ is concentrated on A if and only if $\lambda(A^c) = 0$. Give a counterexample to this statement in the case of a complex measure λ.

60 (Due 2/20). Let f be a real-valued and integrable function on $[0, 1]$. Prove that there exists $c \in [0, 1]$ such that

$$\int_{[0,c]} f\, dm = \int_{[c,1]} f\, dm.$$

61 (Due 2/20). Let $f, f_k \in L^1_m[0, 1]$, for $k = 1, 2, \ldots$. Suppose $f_k(x) \to f(x)$ a.e. on $[0, 1]$, and $\int_{[0,1]} |f_k|\, dm \to \int_{[0,1]} |f|\, dm$. Show that

$$\int_{[0,1]} |f_k - f|\, dm \to 0.$$

Hint: look at $|f| + |f_k| - |f - f_k|$.

62 (Due 2/20). Do the previous exercise with L^1 replaced by L^2.

63 (Due 2/20). Let $X = [0,1]$ and μ the Lebesgue measure. Show that $L^\infty(X)^* \supset L^1(X)$, but $L^\infty(X)^* \neq L^1(X)$ (in the sense $g \to \Phi_g$). (Hint: Use the following consequence of the Hahn-Banach theorem: If X is a Banach space and $A \subset X$ is a closed subspace of X, with $A \neq X$, then there exists $f \in X^*$ with $f \neq 0$, and $f(x) = 0$ for all $x \in A$.)

64 (Due 2/20). Suppose μ is a positive measure on X and $\mu(X) < \infty$. Let $f \in L^\infty_\mu$ and $\|f\|_\infty > 0$. Define

$$\alpha_n = \int_X |f|^n \, d\mu \qquad (n = 1, 2, 3, \dots).$$

Prove that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

(Hint: Start with $\|f\|_\infty = 1$, use Hölder inequality and the result of Exercise 51 (e).)

65 (Due 3/20). Let μ be a complex Borel measure on $\mathbb{R}$ and assume that its symmetric derivative $(D\mu)(x)$ exists at some $x_0 \in \mathbb{R}$. Does it follow that its distribution function $F(x) = \mu((-\infty, x))$ is differentiable at x_0?

66 (Due 3/20). Let $f \in L^1_{\mathbf{m}}(\mathbb{R}^k)$. Show that $|f(x)| \leq (Mf)(x)$ at every Lebesgue point x of f.

67 (Due 3/20). Let

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \quad \text{or} \quad x > 1 \\ 1 & \text{if} \quad 0 < x < 1 \end{cases}.$$

Is it possible to define $f(0)$ and $f(1)$ such that 0 and 1 become Lebesgue points of f?

68 (Bonus). Construct a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and 0 is a Lebesgue point of f, but for every $\varepsilon > 0$

$$\mathbf{m}\big\{x \in \mathbb{R}\colon |x| < \varepsilon \ \text{ and } \ |f(x)| \geq 1\big\} > 0,$$

i.e. f is essentially discontinuous at 0.

69 (Due 3/20). Let $\alpha > 0$, $\beta > 0$ and

$$f(x) = \begin{cases} x^\alpha \cos \frac{\pi}{x^\beta} & \text{for } \ 0 < x \leq 1 \\ 0 & \text{for } \ x = 0 \end{cases}$$

Show that if $\alpha > 1$ and $\alpha > \beta$, then f is absolutely continuous on $[0, 1]$. What about $\alpha \leq \beta$?

70 (Due 3/20). Let f, g be absolutely continuous on $[a, b]$. Show that
(a) fg is absolutely continuous on $[a, b]$.
(b) if $f(x) \neq 0$ for all $x \in [a, b]$, then $1/f$ is absolutely continuous on $[a, b]$.
(c) Does (a) remain true if $[a, b]$ is replaced by $\mathbb{R}$?

71 (Due 3/27). A function $f\colon [a, b] \to \mathbb{C}$ is said to be Lipschitz continuous if $\exists L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in [a, b]$. Prove that if f is Lipschitz continuous, then f is absolutely continuous and $|f'| \leq L$ a.e. Conversely, if f is absolutely continuous and $|f'| \leq L$ a.e., then f is Lipschitz continuous (with that constant L).

72 (Due 3/27). Let $f\colon [a, b] \to \mathbb{R}$ be absolutely continuous. Prove that $V_a^x \leq \int_{[a,x]} |f'| \, d\mathbf{m}$. For an extra credit: is the equality always true?

73 (Due 3/27). Let $f\colon [0, 1] \to \mathbb{R}$ be absolutely continuous on $[\delta, 1]$ for each $\delta > 0$ and continuous and of bounded variation on $[0, 1]$. Prove that f is absolutely continuous on $[0, 1]$. [Hint: use the result of the previous exercise to verify that $f' \in L^1([0, 1])$.]

74 (Bonus). Let $f\colon [a, b] \to \mathbb{C}$ be absolutely continuous. Prove that $|f|$ is absolutely continuous and

$$\left| \frac{d}{dx} |f(x)| \right| \leq |f'(x)|$$

for a.e. $x \in [a, b]$. [Hint: first, use triangle inequality in the form $\big| |p| - |q| \big| \leq |p - q|$, for complex numbers p, q, to show that $|f|$ is AC. Then prove the above inequality for $x \in [a, b]$ that are Lebesgue points for both f' and $|f|'$; use Theorem 10.17.]

75 (Due 4/10). Given measure spaces $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \lambda)$ with σ-finite measures, show that $\mu \times \lambda$ is the unique measure on $\mathfrak{M} \times \mathfrak{N}$ such that

$$(\mu \times \lambda)(A \times B) = \mu(A)\lambda(B)$$

for all measurable rectangles $A \times B$ in $X \times Y$.

76 (Due 4/10). Let $a_n \geq 0$ for $n = 1, 2, \ldots$, and for $t \geq 0$ let

$$N(t) = \#\{n : a_n > t\}$$

Prove that

$$\sum_{n=1}^{\infty} a_n = \int_0^{\infty} N(t)\, dt$$

For an extra credit, find a formula for $\sum_{n=1}^{\infty} \phi(a_n)$ in terms of $N(t)$ as defined above, if $\phi : [0, \infty) \to [0, \infty)$ is locally absolutely continuous (i.e., AC on every finite interval), non-decreasing, and $\phi(0) = 0$ (Hint: use the above fact with $\phi(x) = x$).

77 (Due 4/10). Let $f \in L^1([0, 1])$ and $f \geq 0$. Show that

$$\int_0^1 \frac{f(y)}{|x - y|^{1/2}}\, dy$$

is finite for a.e. $x \in [0, 1]$ and integrable.

78 (Due 4/10). Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^{\infty} e^{-xt}\, dt \quad \text{for} \quad x > 0$$

to prove that

$$\lim_{A \to \infty} \int_0^A \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

79 (Due 4/17). Suppose $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ for some $1 \leq p \leq \infty$. Show that $f * g$ exists at a.e. $x \in \mathbb{R}$ and $f * g \in L^p(\mathbb{R})$, and prove that

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Hints: the case $p = \infty$ is simple and can be treated separately. If $p < \infty$, then use Hölder inequality and argue as in the proof of the previous theorem (see the classnotes).

80 (Due 4/17). Let $f \in L^1(\mathbb{R})$ and

$$g(x) = \int_{\mathbb{R}} f(y)\, e^{-(x-y)^2}\, dy.$$

Show that $g \in L^p(\mathbb{R})$, for all $1 \leq p \leq \infty$, and estimate $\|g\|_p$ in terms of $\|f\|_1$. You can use the following standard fact: $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

81 (Bonus). Let $E = [1, \infty)$ and $f \in L^2_{\mathbf{m}}(E)$. Also assume that $f \geq 0$ a.e. and define

$$g(x) = \int_E f(y)\, e^{-xy}\, dy.$$

Show that $g \in L^1(E)$ and

$$\|g\|_1 \leq c\, \|f\|_2$$

for some $c < 1$. Estimate the minimal value of c the best you can.

82 (Due 4/24). Let $f \colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Prove that

(a) $A = \{(x, y) \in \mathbb{R}^2 \mid y < f(x)\}$ is Lebesgue measurable (in the two-dimensional sense)

(b) Let $f \geq 0$. Is it always true that $\int_{\mathbb{R}} f\, d\mathbf{m}$ equals the Lebesgue measure of A?

(c) $\{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$ is a null set.

83 (Due 4/24). Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be such that f_x is Borel-measurable for every $x \in \mathbb{R}$ and f^y is continuous for every $y \in \mathbb{R}$. Prove that f is Borel-measurable. (See hint on p. 176 in Rudin.)