MA 645-4A (Real Analysis), Dr. Chernov Homework assignment

1 (Due). Show that the open disk $x^2 + y^2 < 1$ is a countable union of planar elementary sets. Show that the closed disk $x^2 + y^2 \leq 1$ is a countable intersection of planar elementary sets.

2 (Due). Prove that every countable set $A \subset X$ is measurable and $\mu(A) = 0$.

3 (Due). Let $A \subset X$ consist of points (x, y) such that either x or y is a rational number. Is A measurable? What is its Lebesgue measure?

4 (Due). Prove that every open set $A \subset X$ is measurable (hint: represent it by a countable union of rectangles). Prove that every closed set $A \subset X$ is measurable.

5 (Due). Let $\mathfrak{L}$ denote the set of all Lebesgue measurable sets (in $\mathbb{R}$). Prove that $\text{card}(\mathfrak{L}) > \mathfrak{C}$, in fact $\text{card}(\mathfrak{L}) = \text{card}(2^{\mathbb{R}})$. (Hint: use the Cantor set and the Cantor theorem).

6 (Due). Let $\mathfrak{U}$ denote the set of all open subsets $U \subset \mathbb{R}$. Prove that $\text{card}(\mathfrak{U}) = \mathfrak{C}$. Do the same for open sets in $\mathbb{R}^2$. Hint: use a countable basis for the respective topology.

7 (Due). Let $X = \{1, 2, 3\}$. Construct all σ-algebras of X.

8 (Due). Let $X = [0, 1]$ and $\mathfrak{G}$ consist of all one-point sets, i.e. $\mathfrak{G} = \big\{\{x\}, x \in X\big\}$. Describe the σ-algebra $\mathfrak{M}(\mathfrak{G})$.

9 (Due). Show that the Borel σ-algebra in $\mathbb{R}$ is generated by the collection of all intervals (r_1, r_2) with rational endpoints $r_1, r_2 \in \mathbb{Q}$.

10 (Due). Show that every Borel set in $\mathbb{R}$ is Lebesgue measurable, but not vice versa.

11 (Bonus). Does there exist an infinite σ-algebra which has only countably many members?

12 (Due). Show that the assumption $\mu(A_1) < \infty$ in Theorem 2.10 is indispensable. Hint: consider the counting measure on $\mathbb{N}$ and take sets $A_n = \{n, n + 1, \ldots\}$.

13 (Due). Let $X \subset \mathbb{R}^2$ be a rectangle. Verify that the collection of all subrectangles $R \subset X$ is a semi-algebra.

14 (Due). Show that if μ is a translation invariant measure defined on the Borel σ-algebra over $\mathbb{R}^2$ such that $\mu(R) < \infty$ for at least one rectangle $R \neq \emptyset$, then there exists a constant $c \geq 0$ such that $\mu(E) = c\mathbf{m}(E)$ for every Borel set E.

15 (Due). Prove that $A \in \mathfrak{M}$ if and only if χ_A is measurable.

16 (Due). Let $(X, \mathfrak{M})$ be a measurable space and $f\colon X \to [-\infty, \infty]$. Prove that f is measurable iff $f^{-1}\big([-\infty, x]\big)$ is a measurable set for every $x \in \mathbb{R}$.

17 (Due). Let $(X, \mathfrak{M})$ be a measurable space and $f\colon X \to [-\infty, \infty]$. Prove that f is measurable iff $f^{-1}\big([-\infty, x]\big)$ is a measurable set for every rational $x \in \mathbb{Q}$.

18 (Due). Let $(X, \mathfrak{M})$ be a measurable space and $f\colon X \to [-\infty, \infty]$ and $g\colon X \to [-\infty, \infty]$ two measurable functions. Prove that the sets

$$\{x\colon f(x) < g(x)\} \qquad \text{and} \qquad \{x\colon f(x) = g(x)\}$$

are measurable.

19 (Due). Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is a Borel function.

20 (Due). Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a Borel function.

21 (Due). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function, i.e. $f(x_1) \leq f(x_2)$ for $x_1 \leq x_2$. Show that f is a Borel function.

22 (Due). Verify that for simple functions the two definitions of the Lebesgue integral given in class agree.

23 (Due). Let $x_0 \in X$ and $\mu = \delta_{x_0}$ the δ-measure. Assume that $\{x_0\} \in \mathfrak{M}$. Show that for every measurable function $f\colon X \to [0,\infty]$ we have

$$\int_X f \, d\mu = f(x_0).$$

24 (Due). Let $X = \mathbb{N}$ and μ the counting measure on the σ-algebra $\mathfrak{M} = 2^{\mathbb{N}}$. Show that for every function $f\colon X \to [0,\infty]$ we have

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} f(n).$$

25 (Due). Let $E \subset X$ be such that $\mu(E) > 0$ and $\mu(E^c) > 0$. Put $f_n = \chi_E$ if n is odd and $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

26 (Due). Construct an example of a sequence of nonnegative measurable functions $f_n\colon X \to [0,\infty)$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists pointwise, but

$$\int_X f \, d\mu < \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

27 (Due). Let $f, g \in L^1(\mu)$ be real-valued functions and $f \le g$. Show that

$$\int_X f \, d\mu \le \int_X g \, d\mu.$$

28 (Due). Let $f_n\colon X \to [0,\infty]$ be a sequence of measurable functions such that $f_1 \ge f_2 \ge \cdots \ge 0$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. Suppose $f_1 \in L^1(\mu)$. Show that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

29 (Due). Let f be a function as above (in the class notes). Fix a $y \in Y$ and define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \setminus N \\ y & \text{if } x \in N \end{cases}.$$

Show that $\tilde{f}\colon X \to Y$ is measurable.

30 (Due). Let f be a function as above and μ a complete measure. Let $g\colon X \to Y$ be an arbitrary (not necessarily measurable) function. Define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \setminus N \\ g(x) & \text{if } x \in N \end{cases}.$$

Show that $\tilde{f}\colon X \to Y$ is measurable.

31 (Due). In the Borel-Cantelli Lemma, let A be the set of points which belong to infinitely many of the sets E_k. Show that

$$A = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k.$$

Use this fact to prove the corollary without any reference to integration.

32 (Due). Suppose $\mu(X) < \infty$. Let $f_n \in L^1_\mu(X)$ be complex measurable functions uniformly converging to a function $f \in L^1_\mu(X)$. Prove that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu. \tag{1}$$

Show that the assumption $\mu(X) < \infty$ cannot be omitted, i.e., give an example of a sequence of functions $f_n \in L^1_\mu(X)$ uniformly converging to a function $f \in L^1_\mu(X)$ such that (1) fails.

33 (Due). Suppose $\mu(X) < \infty$ and f_n are measurable functions defined a.e. on X. Prove that if $f_n \to f$ a.e. on X, then $f_n \to f$ in measure. What happens if $\mu(X) = \infty$?

34 (Due). Prove that if $f_n \to f$ in measure, then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ a.e. on X.

35 (Bonus). Suppose $f \in L^1(\mu)$. Prove that $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\int_E |f| \, d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

36 (Due). (a) Let $f\colon X \to (0, \infty]$ and $\mu(E) > 0$. Show that $\int_E f \, d\mu > 0$.
(b) Let $f, g \in L^1_\mu(X)$ be real-valued functions and $f < g$. Assume $\mu(X) > 0$. Show that

$$\int_E f \, d\mu < \int_E g \, d\mu.$$

37 (Due). Find examples of Lebesgue measurable sets $E_1, E_2 \subset \mathbb{R}$ such that

$$\mu(E_1) < \inf\{\mu(A) \colon E_1 \subset A, \quad A \text{ closed}\}$$

$$\mu(E_2) > \sup\{\mu(V) \colon V \subset E_2, \quad V \text{ open}\}.$$

38 (Due). Let $f \colon \mathbb{R} \to [0, \infty]$ be a Borel function, $f \in L^1(\mu)$, and consider the measure $\rho(E) = \int_E f \, d\mu$, where μ is the Lebesgue measure. Prove that ρ is regular.

39 (Due). Show that the counting measure in $\mathbb{R}^k$ is inner regular, but not outer regular.

40 (Due). Let $s \colon [a, b] \to \mathbb{R}$ be a simple Lebesgue measurable function. Show that for every $\varepsilon > 0$ there is a step function $\varphi \colon [a, b] \to \mathbb{R}$ and a Lebesgue measurable set $E \subset [a, b]$ such that $s(x) = \varphi(x)$ on E and $\mathbf{m}\big([a, b] \setminus E\big) < \varepsilon$. Hint: use the regularity of $\mathbf{m}$.

41 (Due). Let $f \colon [a, b] \to \mathbb{R}$ be a Lebesgue measurable function. Show that for every $\varepsilon > 0$ there is a step function $g \colon [a, b] \to \mathbb{R}$ such that

$$\mathbf{m}\big\{x \in [a, b] \colon |f(x) - g(x)| \geq \varepsilon\big\} < \varepsilon.$$

Hint: use approximation by simple functions and then the previous exercise.

42 (Due). Let $f \in L^1_{\mathbf{m}}(\mathbb{R})$. Prove that there is a sequence $\{g_n\}$ of step functions such that

$$\lim_{n \to \infty} \int_{\mathbb{R}} |f - g_n| \, d\mathbf{m} = 0.$$

Hint: use the previous exercise.

43 (Due). Prove that if $f \colon X \to \mathbb{R}$ is upper (lower) semicontinuous and X is compact, then f is bounded above (below) and attains its maximum (minimum).

44 (Due). Find a function $f(x)$ on $[0, 1]$ such that the improper Riemann integral $\int_0^1 f(x) \, dx$ exists (is finite), but f is not Lebesgue integrable.

45 (Due). Prove that the supremum of any collection of convex functions on (a, b) is convex on (a, b) (assume that the supremum is finite). Prove that a pointwise limit of a sequence of convex functions is convex.

46 (Due). Let φ be convex on (a,b) and ψ convex and nondecreasing on the range of φ. Prove that $\psi \circ \varphi$ is convex on (a,b). For $\varphi > 0$, show that the convexity of $\log \varphi$ implies the convexity of φ, but not vice versa.

47 (Bonus). Assume that φ is a continuous real function on (a,b) such that

$$\varphi\Big(\frac{x+y}{2}\Big) \le \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y)$$

for all $x, y \in (a,b)$. Prove that φ is convex. (The conclusion does not follow if continuity is omitted from the hypotheses.)

48 (Due). Suppose $\mu(X) = 1$ and suppose f and g are two positive measurable functions on X such that $fg \ge 1$. Prove that

$$\int_X f \, d\mu \cdot \int_\Omega g \, d\mu \ge 1.$$

49 (Due). Suppose $\mu(X) = 1$ and $h \colon X \to [0,\infty]$ is measurable. Denote $A = \int_X h \, d\mu$. Prove that

$$\sqrt{1+A^2} \le \int_X \sqrt{1+h^2} \, d\mu \le 1 + A.$$

50 (Bonus). If μ is Lebesgue measure on $[0,1]$ and if h is a continuous function on $[0,1]$ such that $h = f'$, then the inequalities in the previous exercise have a simple geometric interpretation. From this, conjecture (for general X) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

51 (Due). Let $f, g \colon X \to [0,\infty]$. Show that $\text{ess-sup}\,|f+g| \le \text{ess-sup}\,|f| + \text{ess-sup}\,|g|$.

52 (Due). When does one get equality in $\|fg\|_1 \le \|f\|_\infty \, \|g\|_1$?

53 (Due). Suppose $f \colon X \to \mathbb{C}$ is measurable and $\|f\|_\infty > 0$. Define

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \qquad (0 < p < \infty)$$

and consider the set $E = \{p \colon \varphi(p) < \infty\}$. Each of the following questions is graded as a separate exercise. Question (c) and (e) are **bonus** problems.

(a) Let $r < p < s$ and $r, s \in E$. Prove that $p \in E$.

(b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \leq \max\big(\|f\|_r, \|f\|_s\big)$. Show that this implies the inclusion $L^r_\mu \cap L^s_\mu \subset L^p_\mu$.

(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

54 (Due). Let $\mu(X) = 1$. Each of the following questions is graded as a separate exercise.

(a) Prove that $\|f\|_r \leq \|f\|_s$ if $0 < r < s \leq \infty$.

(b) Under what condition does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(c) Prove that $L^r_\mu \supset L^s_\mu$ if $0 < r < s$. If $X = [0, 1]$ and μ is the Lebesgue measure, show that $L^r_\mu \neq L^s_\mu$.

55 (Bonus). For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ if $r \neq s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

56 (Due). (a) Show that $\int_0^{\frac{\pi}{2}} \sqrt{x \sin x}\, dx < \frac{\pi}{2\sqrt{2}}$;
(b) Show that $\left[\int_0^1 x^{\frac{1}{2}}(1-x)^{-\frac{1}{3}}\, dx\right]^3 \leq \frac{8}{5}$.

57 (Bonus). Suppose $1 < p < \infty$ and $f \in L^p((0, \infty))$ relative to the Lebesgue measure. Define

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt \qquad (0 < x < \infty).$$

Prove Hardy's inequality

$$\|F\|_p \leq \frac{p}{p-1} \|f\|_p$$

which shows that the mapping $f \to F$ carries L^p into L^p. [Hint: assume first that $f \geq 0$ and $f \in C_c((0,\infty))$, i.e. the support of f is a finite closed interval $[a,b] \subset (0,\infty)$. Then use integration by parts:

$$\int_{\varepsilon}^{A} F^p(x)\,dx = -p\int_{\varepsilon}^{A} F^{p-1} x F'(x)\,dx$$

where $\varepsilon < a$ and $A > b$. Note that $xF' = f - F$, and apply Hölder inequality to $\int F^{p-1} f\,dx$.]

58 (Due 1/24). Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1_\mu$ define

$$\mu_f(E) = \int_E f\,d\mu.$$

Show:

(a) μ_f is a complex measure;
(b) $|\mu_f| = \mu_{|f|}$, assuming that f is real-valued;
(c) $|\mu_f| = \mu_{|f|}$, now for a general complex-valued $f \in L^1(\mu)$.

59 (Bonus, Due 2/5). Prove that the space $\mathbb{M}(X, \mathfrak{M})$ with the norm $\|\mu\|$ is a Banach space, i.e. it is a complete metric space (every Cauchy sequence converges to a limit). Hint: given a Cauchy sequence of complex measures $\{\mu_n\}$ you need to construct the limit measure μ and prove that $\|\mu_n - \mu\| \to 0$ as $n \to \infty$.

60 (Due 1/31). Let μ be a positive measure on $(X, \mathfrak{M})$ and $f, g \in L^1_\mu$. Prove that

(a) μ_f is concentrated on A if and only if $\mu\{x \in A^c\colon f(x) \neq 0\} = 0$;

(b) $\mu_f \perp \mu_g$ if and only if $\mu\{x \in X\colon f(x)g(x) \neq 0\} = 0$;

(c) if $f \geq 0$, then

$$\mu \ll \mu_f \iff f(x) > 0 \text{ for } \mu - \text{a.e. } x \in X.$$

61 (Due 1/31). Let λ be a positive measure on $(X, \mathfrak{M})$. Prove that λ is concentrated on A if and only if $\lambda(A^c) = 0$. Give a counterexample to this statement in the case of a complex measure λ.

62 (Due 2/5). Let $X = [0, 1]$ and μ the Lebesgue measure on X. Let λ be the counting measure on X. Show that:

(a) λ is not σ-finite.

(b) λ has no Lebesgue decomposition $\lambda_{\mathrm{a}} + \lambda_{\mathrm{s}}$ with respect to μ.

(c) A Lebesgue-measurable function $h \colon X \to \mathbb{C}$ is in L^1_λ if and only if $A \colon = \{x \in X : h(x) \neq 0\}$ is countable, and $\sum_{x \in A} |h(x)| < \infty$. In this case $\int_E h \, d\lambda = \sum_{x \in E \cap A} h(x)$ for all E.

(d) $\mu \ll \lambda$ but there is no $h \in L^1_\lambda$ such that $d\mu = h \, d\lambda$.

63 (Bonus, Due 2/21). Let $X = [0, 1]$ and $\mathbf{m}$ the Lebesgue measure. Show that $L^\infty_{\mathbf{m}}(X)^* \supset L^1_{\mathbf{m}}(X)$, but $L^\infty_{\mathbf{m}}(X)^* \neq L^1_{\mathbf{m}}(X)$ (in the sense $g \to \Phi_g$). (Hint: Use the following consequence of the Hahn-Banach theorem: If X is a Banach space (i.e. a complete metric space, in which every Cauchy sequence converges to a limit) and $A \subset X$ is a closed subspace of X, with $A \neq X$, then there exists $f \in X^*$ with $f \neq 0$, and $f(x) = 0$ for all $x \in A$.)

64 (Due 2/14). Let μ be a complex Borel measure on $\mathbb{R}$ and assume that its symmetric derivative $(D\mu)(x)$ exists at some $x_0 \in \mathbb{R}$. Does it follow that its distribution function $F(x) = \mu((-\infty, x))$ is differentiable at x_0?

65 (Due 2/14). Prove that for all $f \in L^1_{\mathbf{m}}(\mathbb{R}^k)$ and $z > 0$ we have

$$\mathbf{m}(|f| > z) \leq z^{-1} \|f\|_1$$

Conclude that $L^1_{\mathbf{m}}(\mathbb{R}^k) \subset L^1_W(\mathbb{R}^k)$.

66 (Due 2/21). Let $f \in L^1_{\mathbf{m}}(\mathbb{R}^k)$. Show that $|f(x)| \leq (Mf)(x)$ at every Lebesgue point x of f.

67 (Due 2/21). Let

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \quad \text{or} \quad x > 1 \\ 1 & \text{if} \quad 0 < x < 1 \end{cases}$$

Is it possible to define $f(0)$ and $f(1)$ such that 0 and 1 become Lebesgue points of f?

68 (Bonus, Due 2/28). Construct a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and 0 is a Lebesgue point of f, but for every $\varepsilon > 0$

$$\mathbf{m}\big\{x \in \mathbb{R} \colon |x| < \varepsilon \;\text{ and }\; |f(x)| \geq 1\big\} > 0,$$

i.e. f is essentially discontinuous at 0.

69 (Due 2/28). Show that that Cantor function $f\colon [0,1] \to [0,1]$ has the property$f(C) = [0,1]$, i.e., it maps the Cantor set C (which is a null set!) onto the whole interval $[0,1]$.

70 (Due 2/28). Show that

(a) If $f \in C^1([a,b])$, then $V_a^b \leq \int_a^b |f'(x)|\, dx$

(b) If $f \in C([a,b])$ is continuous on $[a,b]$, differentiable on (a,b) and $|f'(x)|$ is bounded on (a,b), then f is of bounded variation on $[a,b]$ (Hint: use the Mean Value Theorem)

71 (Due 2/28). Let

$$f(x) = \begin{cases} x \cos \frac{\pi}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that $V_0^1 = \infty$.

72 (Due 2/28). Let

$$f(x) = \begin{cases} x^2 \cos \frac{\pi}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that $V_0^1 < \infty$, i.e., f is of bounded variation on $[0,1]$.

73 (Due 2/28). Let f, g be absolutely continuous on $[a,b]$. Show that

(a) $f \pm g$ are absolutely continuous on $[a,b]$

(b) fg is absolutely continuous on $[a,b]$

(c) if $f(x) \neq 0$ for all $x \in [a,b]$, then $1/f$ is absolutely continuous on $[a,b]$

74 (Due 2/28). A function $f\colon [a,b] \to \mathbb{C}$ is said to be Lipschitz continuous if $\exists L > 0$ such that $|f(x) - f(y)| \le L|x - y|$ for all $x, y \in [a,b]$. Prove that if f is Lipschitz continuous, then f is absolutely continuous and $|f'| \le L$ a.e. Conversely, if f is absolutely continuous and $|f'| \le L$ a.e., then f is Lipschitz continuous (with that constant L).

75 (Due 2/28). Let

$$f(x) = \begin{cases} x^2 \cos \frac{\pi}{x^2} & \text{if } \ 0 < x \le 1 \\ 0 & \text{if } \ x = 0 \end{cases}$$

(a) Show that $f(x)$ is differentiable at every point $x \in [0,1]$ (including $x = 0$)

(b) Verify that $f'(x)$ *does not* belong to $L^1([0,1])$

Conclude that f is not absolutely continuous.

76 (Due 2/28). Let

$$f(x) = \begin{cases} x^2 \cos \frac{\pi}{x} & \text{if } \ 0 < x \le 1 \\ 0 & \text{if } \ x = 0 \end{cases}$$

(a) Show that $f(x)$ is differentiable at every point $x \in [0,1]$ (including $x = 0$)

(b) Verify that $f'(x)$ *does* belong to $L^1([0,1])$

Conclude that f is absolutely continuous.

77 (Due 3/7, the bonus part is due on 3/14). Let $f\colon [a,b] \to \mathbb{R}$ be absolutely continuous. Prove that $V_a^x \le \int_{[a,x]} |f'| d\mathbf{m}$.
Bonus part: is the equality always true?

78 (Bonus, Due 3/14). Let $f\colon [0,1] \to \mathbb{R}$ be absolutely continuous on $[\delta, 1]$ for each $\delta > 0$, continuous at $x = 0$, and of bounded variation on $[0,1]$. Prove that f is absolutely continuous on $[0,1]$. (Note: you can use the bonus part of the previous exercise only if you properly finish it first.)

79 (Due 3/14). Let $f \ge 0$ and $f \in L^1(\mathbb{R})$. Find

$$\lim_{n\to\infty} \int_{[0,1]} f(nx)\, d\mathbf{m}(x)$$

80 (Due 3/14). Let $f, g\colon [a,b] \to \mathbb{C}$ be two AC functions. Prove the integration-by-parts formula

$$\int_{[a,b]} f' g \, d\mathbf{m} = f(b)g(b) - f(a)g(a) - \int_{[a,b]} f g' \, d\mathbf{m}$$

81 (Due 3/14). Let $f \geq 0$ and $f \in L^1([0,\infty))$ and $g(x) \colon = 2xf(x^2)$ for all $x \in [0,\infty)$. Show that $g \in L^1([0,\infty))$ and

$$\int_{[0,\infty)} f \, d\mathbf{m} = \int_{[0,\infty)} g \, d\mathbf{m}$$

82 (Bonus, Due 3/28). Let $f\colon \mathbb{R} \to \mathbb{R}$ be integrable, i.e., $f \in L^1_{\mathbf{m}}(\mathbb{R})$. Define a function $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f\left(x - \frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that $g(-1/x) = g(x)$ and

$$\int_{\mathbb{R}} f \, d\mathbf{m} = \int_{\mathbb{R}} g \, d\mathbf{m}$$

83 (Due 3/28). Given measure spaces $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \lambda)$ with σ-finite positive measures, show that $\mu \times \lambda$ is the unique measure on $\mathfrak{M} \times \mathfrak{N}$ such that

$$(\mu \times \lambda)(A \times B) = \mu(A)\lambda(B)$$

for all measurable rectangles $A \times B$ in $X \times Y$.

84 (Due 4/11). Let $a_n \geq 0$ for $n = 1, 2, \ldots$, and for $t \geq 0$ let

$$N(t) = \#\{n : a_n > t\}$$

Prove that

$$\sum_{n=1}^{\infty} a_n = \int_0^{\infty} N(t) \, dt$$

85 (Bonus, Due 4/11). Generalize the previous exercise as follows. Let $\phi\colon [0,\infty) \to [0,\infty)$ be an increasing locally absolutely continuous function (the latter means that ϕ is AC on every finite interval) such that $\phi(0) = 0$. Find a formula for $\sum_{n=1}^{\infty} \phi(a_n)$ in terms of $N(t)$.

86 (Due 4/11). Let $f \in L^1([0,1])$ and $f \geq 0$. Show that

$$\int_0^1 \frac{f(y)}{|x-y|^{1/2}}\,dy$$

is finite for a.e. $x \in [0,1]$ and, as a function of x, integrable with respect to the Lebesgue measure on $[0,1]$.

87 (Due 4/11). Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt}\,dt \quad \text{for} \quad x > 0$$

to prove that

$$\lim_{A\to\infty} \int_0^A \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

88 (Due 4/11). Let E be a Lebesgue measurable subset of $\mathbb{R}^2$. Suppose that for a.e. $x \in \mathbb{R}$ the set $E_x = \{y \in \mathbb{R} \colon (x,y) \in E\}$ has Lebesgue measure zero. Prove that for a.e. $y \in \mathbb{R}$ the set $E^y = \{x \in \mathbb{R} \colon (x,y) \in E\}$ has Lebesgue measure zero. Compute $\mathbf{m}_2(E)$.

89 (Due 4/18). Suppose $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ for some $1 \leq p \leq \infty$. Show that $f * g$ exists at a.e. $x \in \mathbb{R}$ and $f * g \in L^p(\mathbb{R})$, and prove that

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Hints: the case $p = \infty$ is simple and can be treated separately. If $p < \infty$, then use Hölder inequality and argue as in the proof of Theorem 18.3.

90 (Due 4/18). Let $f \in L^1(\mathbb{R})$ and

$$g(x) = \int_{\mathbb{R}} f(y)\, e^{-(x-y)^2}\, d\mathbf{m}(y).$$

Show that $g \in L^p(\mathbb{R})$, for all $1 \leq p \leq \infty$, and estimate $\|g\|_p$ in terms of $\|f\|_1$. You can use the following standard fact: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

91 (Bonus, Due 4/18). Let $E = [1,\infty)$ and $f \in L^2_{\mathbf{m}}(E)$. Also assume that $f \geq 0$ a.e. and define

$$g(x) = \int_E f(y)\, e^{-xy}\, d\mathbf{m}(y).$$

Show that $g \in L^1(E)$ and

$$\|g\|_1 \leq c\,\|f\|_2$$

for some $c < 1$. Estimate the minimal value of c the best you can.

92 (Due 4/18). Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable.

(a) Prove that the set

$$A = \{(x, y) \in \mathbb{R}^2 \mid y < f(x)\}$$

is Lebesgue measurable (in the two-dimensional sense)

(b) Let $f \geq 0$. Is it always true that $\int_{\mathbb{R}} f \, d\mathbf{m}$ equals the Lebesgue measure of A?

(c) Prove that $\{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$ is a null set.

93 (Bonus, Due 4/18). Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be such that f_x is Borel-measurable for every $x \in \mathbb{R}$ and f^y is continuous for every $y \in \mathbb{R}$. Prove that f is Borel-measurable. (See hint on p. 176 in Rudin's book.)