Real Analysis

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1 Lebesgue measure in $\mathbb{R}^2$

DEFINITION 1.1. An **interval** $I \subset \mathbb{R}$ is a set of the form $[a, b]$ or $[a, b)$ or $(a, b]$ or (a, b), where $a \leq b$ are real numbers. Its length is $|I| = b - a$.

DEFINITION 1.2. A **rectangle** $R \subset \mathbb{R}^2$ is a set of the form $R = I_1 \times I_2$, where I_1, I_2 are intervals. The area of a rectangle is $\mathrm{Area}(R) = |I_1| \times |I_2|$.

DEFINITION 1.3. An **elementary set** is a finite union of disjoint rectangles, i.e. $B = \cup_{i=1}^n R_i$, where $R_i \cap R_j = \emptyset$ for $i \neq j$. The area of an elementary set is $\mathrm{Area}(B) = \sum_{i=1}^n \mathrm{Area}(R_i)$.

Theorem 1.4. *The area of an elementary set does not depend on how it is partitioned into disjoint rectangles.*

Theorem 1.5. *Finite unions, intersections and differences of elementary sets are elementary sets. Thus, elementary sets make a ring, see below.*

DEFINITION 1.6. A **ring** is a nonempty collection of subsets of a set X closed under finite unions, intersections and differences.

DEFINITION 1.7. An **algebra** is a ring containing X itself.

From now on we will work with sets $A \subset X$ within a fixed rectangle $X \subset \mathbb{R}^2$. Elementary sets in X make an algebra.

DEFINITION 1.8. The **outer measure** of a set $A \subset X$ is $\mu^*(A) = \inf \sum_{i=1}^\infty \mathrm{Area}(R_i)$ where the infimum is taken over all countable covers of A by rectangles, $A \subset \cup_{i=1}^\infty R_i$.

DEFINITION 1.9. The **inner measure** of a set $A \subset X$ is $\mu_*(A) = \mathrm{Area}(X) - \mu^*(A^c)$ where $A^c = X \setminus A$ is the complement of A.

Lemma. *For any elementary set B we have $\mu^*(B) = \mathrm{Area}(B)$.*

Proof: BWOC, suppose $\mu^*(B) < \mathrm{Area}(B)$. If B is a closed set covered by open rectangles, then due to compactness there is a finite subcover. If B is not closed, shrink it a bit to make it closed. If the rectangles are not open, expand them a bit to make them open.

Theorem 1.10. *For any set $A \subset X$ we have* $0 \le \mu_*(A) \le \mu^*(A) \le \text{Area}(X)$.

Note: $\mu_*(A) \le \mu^*(A)$ is equivalent to $\mu^*(A) + \mu^*(A^c) \ge \text{Area}(X)$, which follows from the previous lemma.

DEFINITION 1.11. A set $A \subset X$ is **measurable** if $\mu_*(A) = \mu^*(A)$; then its **Lebesgue measure** is defined by $\mathbf{m}(A) = \mu_*(A) = \mu^*(A)$.

Note: a set $A \subset X$ is measurable if and only if $\mu^*(A)+\mu^*(A^c) = \text{Area}(X)$. This implies that if A measurable, then A^c is measurable, too.

Note: Let $\cup_i R_i = C_1 \supset A$ and $\cup_j R'_j = C_2 \supset A^c$ be countable covers of A and A^c by rectangles, respectively. We can choose them so that $\sum_i \text{Area}(R_i) + \sum_j \text{Area}(R'_j) < \text{Area}(X) + \varepsilon$. Then $\sum_{i,j} \text{Area}(R_i \cap R_j) < \varepsilon$.

Note: an empty set is measurable and $\mathbf{m}(\emptyset) = 0$.

EXERCISE 1. Prove that every countable set $A \subset X$ is measurable and $\mathbf{m}(A) = 0$.

EXERCISE 2 (Bonus). Let $A \subset X$ consist of points (x, y) such that either x or y is a rational number. Is A measurable? What is its Lebesgue measure?

Theorem 1.12. *The Lebesgue measure is translationally invariant, i.e. if $A \subset X$ is measurable and $a \in \mathbb{R}^2$, then the set*

$$A + a = \{x + a \colon x \in A\}$$

is measurable and $\mathbf{m}(A + a) = \mathbf{m}(A)$.

The Lebesgue measure can be constructed in $\mathbb{R}$ by using intervals (and their length) instead of rectangles (and area). The Lebesgue measure in $\mathbb{R}$ is translationally invariant, too, i.e. if A is measurable and $a \in \mathbb{R}$, then the set $A + a = \{x + a \colon x \in A\}$ is also measurable and $\mathbf{m}(A + a) = \mathbf{m}(A)$.

The Lebesgue measure can be extended to unbounded sets $A \subset \mathbb{R}^2$ as follows: A is measurable if and only if $A \cap X$ is measurable for every rectangle $X \subset \mathbb{R}^2$. Its measure is

$$\mathbf{m}(A) = \lim_{n\to\infty} \mathbf{m}(A \cap X_n),$$

where $X_n = [-n, n] \times [-n, n]$ is a growing sequence of rectangles. The value of $\mathbf{m}(A)$ does not depend on the particular growing sequence of rectangles.

Theorem 1.13 (σ-subadditivity). *If $A \subset \cup_{i=1}^{\infty} A_i$, then $\mu^*(A) \le \sum_{i=1}^{\infty} \mu^*(A_i)$.*

Corollary 1.14. *If $A_1 \subset A_2$, then $\mu^*(A_1) \leq \mu^*(A_2)$.*

Theorem 1.15. *Every elementary set $B \subset X$ is measurable and its Lebesgue measure is equal to its area, i.e. $\mathbf{m}(B) = \text{Area}(B)$.*

Lemma 1.16. *For any two sets $A_1, A_2 \subset X$ we have $|\mu^*(A_1) - \mu^*(A_2)| \leq \mu^*(A_1 \Delta A_2)$.*

Proof: BWOC, suppose $\mu^*(A_1) > \mu^*(A_2) + \mu^*(A_1 \Delta A_2)$, then $\mu^*(A_1) > \mu^*(A_2) + \mu^*(A_1 \setminus A_2)$, which contradicts Theorem 1.13

Theorem 1.17 (Approximation). *A set $A \subset X$ is measurable if and only if for any $\varepsilon > 0$ there exists an elementary set $B \subset X$ such that $\mu^*(A \Delta B) < \varepsilon$.*

Note that, in the context of the above theorem, $|\mathbf{m}(A) - \text{Area}(B)| \leq \varepsilon$.

Proof: "$\Rightarrow$": Let $\cup_i R_i = C_1 \supset A$ and $\cup_j R'_j = C_2 \supset A^c$ be countable covers of A and A^c by rectangles, respectively, such that $\sum_i \text{Area}(R_i) + \sum_j \text{Area}(R'_j) < \text{Area}(X) + \varepsilon$. Define $B = \cup_{i=1}^N R_i$ so that $\sum_{i=N+1}^\infty \text{Area}(R_i) < \varepsilon$. Then $A \Delta B \subset (C_1 \setminus B) \cup (B \cap C_2)$, and $\mu^*(C_1 \setminus B) < \varepsilon$ by construction, while $\mu^*(B \cap C_2) < \varepsilon$ according to the second remark after Definition 1.11.

"$\Leftarrow$": Note that $\mu^*(A^c \Delta B^c) = \mu^*(A \Delta B) < \varepsilon$, hence by the previous lemma $\mu^*(A) \approx \mu^*(B) = \text{Area}(B)$ and $\mu^*(A^c) \approx \mu^*(B^c) = \text{Area}(B^c)$ (here $\approx$ means that the two measures are ε-close). Lastly, note that $\text{Area}(B) + \text{Area}(B^c) = \text{Area}(X)$, hence $\mu^*(A) + \mu^*(A^c) \approx \text{Area}(X)$.

Theorem 1.18. *Finite unions, intersections and differences of measurable sets are measurable.*

Theorem 1.19 (Additivity). *If $A_1, \dots, A_n$ are disjoint measurable sets, then $\mathbf{m}\big(\cup_{i=1}^n A_i\big) = \sum_{i=1}^n \mathbf{m}(A_i)$.*

Theorem 1.20. *Countable unions and intersections of measurable sets are measurable. Thus measurable sets make a σ-algebra, see below.*

Definition 1.21. A **σ-algebra** is an algebra closed under countable unions and intersections. (It is enough to mentions countable unions.)

Theorem 1.22 (σ-additivity). *If $\{A_i\}_{i=1}^\infty$ is a countable collection of pairwise disjoint measurable sets, then $\mathbf{m}\big(\cup_{i=1}^\infty A_i\big) = \sum_{i=1}^\infty \mathbf{m}(A_i)$.*

Exercise 3. Prove that every open set $A \subset X$ is measurable (hint: represent it by a countable union of rectangles). Prove that every closed set $A \subset X$ is measurable.

Note: every G_δ set is measurable and every F_σ set is measurable. (Recall: G_δ set is a countable intersection of open sets; F_σ set is a countable union of closed sets.)

Note: every set obtained from open or closed sets by a sequence countable unions, intersections, differences, etc., is a measurable set.

Theorem 1.23. *If $A \subset X$ is measurable and $\mathbf{m}(A) = 0$, then every subset $A' \subset A$ is measurable and $\mathbf{m}(A') = 0$.*

Theorem 1.24. *If $A \subset X$ is measurable and $\mathbf{m}(A) > 0$, then there is a non-measurable subset $A' \subset A$.*

Theorem 1.25 (Continuity - I)**.** *Let $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ be a sequence of measurable sets (called* monotonically decreasing sequence*). Then $\lim_{n\to\infty} \mathbf{m}(A_n) = \mathbf{m}(A)$, where $A = \cap_{n=1}^{\infty} A_n$.*

Theorem 1.26 (Continuity - II)**.** *Let $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ be a sequence of measurable sets (called* monotonically increasing sequence*). Then $\lim_{n\to\infty} \mathbf{m}(A_n) = \mathbf{m}(A)$, where $A = \cup_{n=1}^{\infty} A_n$.*

Example: the middle-third Cantor set $F \subset [0, 1]$ is a closed uncountable set of zero measure. Its cardinality is the same as that of $\mathbb{R}$.

DEFINITION 1.27. Two sets A and B have the same **cardinality**, $\operatorname{card}(A) = \operatorname{card}(B)$, if there is a bijection $\varphi\colon A \leftrightarrow B$. The set A has cardinality smaller than that of B, $\operatorname{card}(A) < \operatorname{card}(B)$, if there is an injection $\varphi\colon A \to B$ but not vice versa.

Note: 'same cardinality' is an equivalence relations. Thus all sets of the same cardinality make an (equivalence) class. For any two sets A and B there is either an injection $A \to B$ or an injection $B \to A$ (this follows from a 'well ordering theorem', which we do not quote here).

Theorem 1.28 (Cantor-Bernstein-Schroeder)**.** *If there is an injection $A \to B$ and an injection $B \to A$, then there is a bijection $A \leftrightarrow B$. Hence, for any two sets A and B there are three possibilities:* $\operatorname{card}(A) = \operatorname{card}(B)$ *or* $\operatorname{card}(A) > \operatorname{card}(B)$ *or* $\operatorname{card}(A) < \operatorname{card}(B)$*.*

For a finite set A of n elements, we simply put $\operatorname{card}(A) = n$. For the set of integers $\mathbb{N}$, we put $\operatorname{card}(\mathbb{N}) = \aleph_0$ ('aleph-null'). Then any countable set has cardinality $\aleph_0$, in particular, $\operatorname{card}(\mathbb{Z}) = \operatorname{card}(\mathbb{Q}) = \aleph_0$.

The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{C}$ and is called **continuum**. We have $\text{card}(I) = \mathfrak{C}$ for any interval $I \subset \mathbb{R}$ of positive length. We have $\text{card}(\mathbb{R}^n) = \mathfrak{C}$ for every $n \geq 1$. We also have $\aleph_0 < \mathfrak{C}$.

The **continuum hypothesis** states that there is no set A such that $\aleph_0 < \text{card}(A) < \mathfrak{C}$. In particular, every subset of $\mathbb{R}$ is either finite or countable or has the same cardinality as $\mathbb{R}$. The continuum hypothesis cannot be proved or disproved if one uses standard mathematical axioms. Thus it may (or may not) be adopted as an independent axiom.

DEFINITION 1.29. For any set A, the **power set** (or **powerset**) of A, denoted by 2^A, is the set of all subsets of A.

Note: if $\text{card}(A) = n < \infty$, then $\text{card}(2^A) = 2^n$.

Theorem 1.30 (Cantor)**.** *For any set A we have* $\text{card}(A) < \text{card}(2^A)$.

The set $2^{\mathbb{N}}$ can be identified with the set of all infinite sequences of zeros and ones. This set has cardinality $\mathfrak{C}$.

EXERCISE 4. Let $\mathfrak{L}$ denote the set of all Lebesgue measurable sets (in $\mathbb{R}$). Prove that $\text{card}(\mathfrak{L}) > \mathfrak{C}$.

EXERCISE 5. Let $\mathfrak{U}$ denote the set of all open subsets $U \subset \mathbb{R}$. Prove that $\text{card}(\mathfrak{U}) = \mathfrak{C}$. Do the same for open sets in $\mathbb{R}^2$.

2 General measures

General measures are defined on sets with σ-algebras.

DEFINITION 2.1. A set X with a σ-algebra $\mathfrak{M}$ of its subsets is called a **measurable space**. Sets $A \in \mathfrak{M}$ are said to be **measurable**.

DEFINITION 2.2. Let $(X, \mathfrak{M})$ be a measurable space. A **measure** is a function μ, defined on $\mathfrak{M}$, whose range is $[0, \infty]$ and which is σ-additive. To avoid trivialities, we shall also assume that $\mu(A) < \infty$ for at least one $A \in \mathfrak{M}$. A measurable space with a measure is called a **measure space**.

For any set X, there are two trivial σ-algebras. One is *minimal*, it consists of the sets X and $\emptyset$ only. The other is *maximal*, it contains all the subsets of X. The latter one is denoted by 2^X.

EXERCISE 6. Let $X = \{1, 2, 3\}$. Construct all σ-algebras of X.

If X is finite or countable, we will always use the maximal σ-algebra 2^X. In that case every measure μ on $(X, 2^X)$ is determined by its values $p_x = \mu(\{x\})$ on one-point sets $\{x\}, x \in X$.

Theorem 2.3. *Let $\{\mathfrak{M}_\alpha\}$ be an arbitrary collection of σ-algebras of a set X. Then their intersection $\cap_\alpha \mathfrak{M}_\alpha$ is a σ-algebra of X as well.*

Theorem 2.4. *Let $\mathfrak{G}$ be any collection of subsets of X. Then there exists a unique σ-algebra $\mathfrak{M}^* \supset \mathfrak{G}$ such that for any other σ-algebra $\mathfrak{M} \supset \mathfrak{G}$ we have $\mathfrak{M}^* \subset \mathfrak{M}$. (In other words, $\mathfrak{M}^*$ is the minimal σ-algebra containing $\mathfrak{G}$.)*

DEFINITION 2.5. We say that the minimal σ-algebra $\mathfrak{M}^*$ containing the given collection $\mathfrak{G}$ is **generated** by $\mathfrak{G}$. We also denote it by $\mathfrak{M}(\mathfrak{G})$.

EXERCISE 7. Let $X = [0, 1]$ and $\mathfrak{G}$ consist of all one-point sets, i.e. $\mathfrak{G} = \big\{\{x\}, x \in X\big\}$. Describe the σ-algebra $\mathfrak{M}(\mathfrak{G})$.

It is interesting to compare the notion of σ-algebra with topology. The standard topology in $\mathbb{R}$ is not a σ-algebra. On the other hand, the σ-algebra in the previous exercise is not a topology.

DEFINITION 2.6. Let X be a topological space. The σ-algebra $\mathfrak{M}$ generated by the collection of all open subsets $U \subset X$ is called the **Borel** σ-algebra. Its members are called **Borel sets**.

Note: all closed sets, all F_σ sets and all G_δ sets are Borel sets. The cardinality of the Borel σ-algebra of $\mathbb{R}$ is $\mathfrak{C}$, i.e. continuum.

EXERCISE 8. Show that every Borel set in $\mathbb{R}$ is Lebesgue measurable, but not vice versa.

EXERCISE 9 (Bonus). Does there exist an infinite σ-algebra which has only countably many members?

DEFINITION 2.7. Let $(X, \mathfrak{M})$ be a measurable space. For any $X \in \mathfrak{M}$, define $\mu(A) = \infty$ if A is an infinite set, and $\mu(A) = \operatorname{card}(A)$ if A is finite. Then μ is called **counting measure**.

DEFINITION 2.8. Let $(X, \mathfrak{M})$ be a measurable space and $x \in X$. The measure δ_x defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

is called the **delta-measure** or the **Dirac measure** (concentrated at x).

Theorem 2.9. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then $\mu(\emptyset) = 0$ and for every measurable sets $A \subset B$ we have $\mu(A) \leq \mu(B)$.*

Theorem 2.10 (Continuity - I)**.** *Let $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ be a sequence of measurable sets, and $\mu(A_1) < \infty$. Then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$, where $A = \cap_{n=1}^{\infty} A_n$.*

Theorem 2.11 (Continuity - II)**.** *Let $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ be a sequence of measurable sets. Then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$, where $A = \cup_{n=1}^{\infty} A_n$.*

EXERCISE 10. Show that the assumption $\mu(A_1) < \infty$ in Theorem 2.10 is indispensable. Hint: consider the counting measure on $\mathbb{N}$ and take sets $A_n = \{n, n+1, \ldots\}$.

In many cases σ-algebras tend to be rather large, an explicit definition (or description) of $\mu(A)$ for all $A \in \mathfrak{M}$ is often an impossible task. It is common to define $\mu(A)$ on a smaller collection of sets, $\mathfrak{E}$, and extend it to $\mathfrak{M}(\mathfrak{E})$.

DEFINITION 2.12. A **semi-algebra** is a nonempty collection $\mathfrak{E}$ of subsets of X with two properties: (i) it is closed under intersections; i.e. if $A, B \in \mathfrak{E}$, then $A \cap B \in \mathfrak{E}$; and (ii) if $A \in \mathfrak{E}$, then $A^c = \cup_{i=1}^{n} A_i$, where each $A_i \in \mathfrak{E}$ and $A_1, \ldots, A_n$ are pairwise disjoint subsets of X.

EXERCISE 11. Let $X \subset \mathbb{R}^2$ be a rectangle. Verify that the collection of all subrectangles $R \subset X$ is a semi-algebra.

Theorem 2.13 (Extension)**.** *Let $\mathfrak{E}$ be a semi-algebra of X. Let ν be a function on $\mathfrak{E}$, whose range is $[0, \infty)$ and which is σ-additive, i.e. for any $A \in \mathfrak{E}$ such that $A = \cup_{i=1}^{\infty} A_i$ and $A_i \in \mathfrak{E}$ are pairwise disjoint sets, we have $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$. Then there is a unique measure μ on the σ-algebra $\mathfrak{M}(\mathfrak{E})$ that agrees with ν on $\mathfrak{E}$, i.e. $\mu(A) = \nu(A)$ for all $A \in \mathfrak{E}$.*

We accept this theorem without proof. Its proof is basically the repetition of our construction of the Lebesgue measure in $\mathbb{R}^2$.

Corollary 2.14. *Let $X \subset \mathbb{R}^2$ be a rectangle. There is a unique measure μ on the Borel σ-algebra of X such that for every rectangle $R \subset X$ we have $\mu(R) = m(R)$, where $m(R)$ is the the area of R.*

The following useful theorem is also given without proof:

Theorem 2.15. *Let $(X, \mathfrak{M})$ be a measurable space and $\mathfrak{E}$ a collection of subsets of X that generates $\mathfrak{M}$, i.e. such that $\mathfrak{M}(\mathfrak{E}) = \mathfrak{M}$. Suppose two measures, μ_1 and μ_2, agree on $\mathfrak{E}$, i.e. $\mu_1(A) = \mu_2(A)$ for all $A \in \mathfrak{E}$, and $\mu_1(X) = \mu_2(X)$. Then $\mu_1 = \mu_2$.*

The extension theorem provides a measure on the minimal σ-algebra containing the given semi-algebra. It is often convenient to complete it in the way we constructed the Lebesgue measure.

Definition 2.16. A measure μ on a measurable space $(X, \mathfrak{M})$ is said to be **complete** if every subset of any set of measure zero is measurable, i.e. if $B \subset A$ and $\mu(A) = 0$ imply $B \in \mathfrak{M}$ (and then obviously $\mu(B) = 0$).

Sets of zero measure are called **null sets**. Their complements are called **full measure sets**.

Theorem 2.17 (Completion)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then*

$$\bar{\mathfrak{M}} = \{A \cup E \colon A \in \mathfrak{M}, \quad E \subset N \text{ for some null set } N\}$$

is a σ-algebra. For every $B = A \cup E$ as above define $\bar{\mu}(B) = \mu(A)$. Then $\bar{\mu}$ is a complete measure on $(X, \bar{\mathfrak{M}})$. Moreover, if μ^* is another complete measure that coincides with μ on $\mathfrak{M}$, then μ^* coincides with $\bar{\mu}$ on $\bar{\mathfrak{M}}$.

$\bar{\mu}$ is called the **completion** of μ. The completion of the measure constructed in Corollary 2.14 is exactly the Lebesgue measure on $X \subset \mathbb{R}^2$.

3 Measurable functions

Recall: given two topological spaces X and Y, a function $f \colon X \to Y$ is continuous iff for any open set $V \subset Y$ its preimage $f^{-1}(V)$ is open, too.

Definition 3.1. Let $(X, \mathfrak{M})$ be a measurable space and Y a topological space. A function $f \colon X \to Y$ is a **measurable function** iff for any open set $V \subset Y$ its preimage is measurable, i.e. $f^{-1}(V) \in \mathfrak{M}$.

The most interesting functions for us are real-valued ($Y = \mathbb{R}$) and complex-valued ($Y = \mathbb{C}$). We also consider the **extended real line** $[-\infty, \infty]$ with topology in which $(a, \infty]$ and $[-\infty, b)$ are open sets. Then $f \colon X \to [-\infty, \infty]$ is an **extended real-valued function**.

Theorem 3.2. *Let $(X, \mathfrak{M})$ be a measurable space and Y, Z topological spaces. If $f \colon X \to Y$ is measurable and $g \colon Y \to Z$ is continuous, then their composition $g \circ f \colon X \to Z$ is measurable.*

Recall that the Borel σ-algebra is generated by open sets.

DEFINITION 3.3. Let X, Y be topological spaces. A function $f\colon X \to Y$ is a **Borel function** iff for any open set $V \subset Y$ its preimage is a Borel set.

Theorem 3.4. *Continuous functions are Borel functions.*

There are many more Borel functions than continuous functions.

Theorem 3.5. *Let $(X, \mathfrak{M})$ be a measurable space and $f\colon X \to Y$ a function. Then*

(a) $\mathfrak{G} = \{E \subset Y\colon f^{-1}(Y) \in \mathfrak{M}\}$ *is a σ-algebra in Y;*

(b) *if Y is a topological space and f measurable, then $f^{-1}(E) \in \mathfrak{M}$ for any Borel set $E \subset Y$;*

(c) *If Y and Z are topological spaces, $f\colon X \to Y$ is measurable and $g\colon Y \to Z$ is a Borel function, then $g \circ f\colon X \to Z$ is measurable.*

Given a subset $A \subset X$ the function

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is called the **characteristic function** of A (or the **indicator** of A).

EXERCISE 12. Prove that $A \in \mathfrak{M}$ if and only if χ_A is measurable.

Theorem 3.6. *Let $(X, \mathfrak{M})$ be a measurable space and $f\colon X \to [-\infty, \infty]$. Then f is measurable iff $f^{-1}\big([-\infty, x)\big)$ is a measurable set for every $x \in \mathbb{R}$.*

EXERCISE 13. In the context of the previous theorem, prove that f is measurable iff $f^{-1}\big([-\infty, x]\big)$ is a measurable set for every $x \in \mathbb{R}$.

EXERCISE 14. In the context of the previous theorem, prove that f is measurable iff $f^{-1}\big([-\infty, x)\big)$ is a measurable set for every rational $x \in \mathbb{Q}$.

EXERCISE 15. Let $(X, \mathfrak{M})$ be a measurable space and $f\colon X \to [-\infty, \infty]$ and $g\colon X \to [-\infty, \infty]$ two measurable functions. Prove that the sets

$$\{x\colon f(x) < g(x)\} \qquad \text{and} \qquad \{x\colon f(x) = g(x)\}$$

are measurable.

Theorem 3.7. *Let $(X, \mathfrak{M})$ be a measurable space and $u\colon X \to \mathbb{R}$ and $v\colon X \to \mathbb{R}$ two measurable functions. Let Y be a topological space and $\Phi\colon \mathbb{R}^2 \to Y$ a continuous function. Then $h(x) = \Phi(u(x), v(x))$ is a measurable function from X to Y.*

Theorem 3.8. *Let* $(X, \mathfrak{M})$ *be a measurable space and* $u\colon X \to \mathbb{R}$ *and* $v\colon X \to \mathbb{R}$ *two functions. Then* $f(x) = u(x) + \mathbf{i}v(x)$ *is a measurable function from* X *to* $\mathbb{C}$ *if and only if both* $u(x)$ *and* $v(x)$ *are measurable functions.*

Theorem 3.9. *Let* $(X, \mathfrak{M})$ *be a measurable space and* $f\colon X \to \mathbb{C}$ *and* $g\colon X \to \mathbb{C}$ *two measurable functions. Then* $|f|$, $f+g$, $f-g$, *and* fg *are measurable functions.*

Theorem 3.10 (Polar factorization)**.** *Let* $(X, \mathfrak{M})$ *be a measurable space and* $f\colon X \to \mathbb{C}$ *a measurable function. Then* $f = g|f|$, *where* $g\colon X \to \mathbb{C}$ *is a measurable function such that* $|g| = 1$.

Note: arithmetic operations in $[-\infty, \infty]$ are defined in an obvious way (for example, $a + \infty = \infty$ for any $a \in (-\infty, \infty]$, $a + (-\infty) = -\infty$ for any $a \in [-\infty, \infty)$, $a \cdot \infty = \infty$ for every $a \in (0, \infty]$, etc.; we also put $0 \cdot \infty = 0$, a bit arbitrarily; the sum $\infty + (-\infty)$ is not defined). The set $[-\infty, \infty]$ is naturally ordered, any subset $A \subset [-\infty, \infty]$ obviously has $\inf A$ a $\sup A$. The convergence of sequences in $[-\infty, \infty]$ is defined by using its topology. For any sequence $\{a_n\}$ in $[-\infty, \infty]$ we naturally define $\liminf a_n$ and $\limsup a_n$:

$$\limsup_{n\to\infty} a_n = \inf\{b_1, b_2, \ldots\}, \qquad b_n = \sup\{a_n, a_{n+1}, \ldots\}.$$

Theorem 3.11. *Let* $(X, \mathfrak{M})$ *be a measurable space and* $f_n\colon X \to [-\infty, \infty]$ *measurable functions. Then*

$$g = \sup_{n\geq 1} f_n \qquad \text{and} \qquad h = \limsup_{n\to\infty} f_n$$

are measurable functions. (Similarly for $\inf f_n$ *and* $\liminf f_n$*.)*

Corollary 3.12. *Let* $(X, \mathfrak{M})$ *be a measurable space and* $f_n\colon X \to [-\infty, \infty]$ *measurable functions. If the limit*

$$g(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$, *then* g *is a measurable function.*

Corollary 3.13. *Let* $(X, \mathfrak{M})$ *be a measurable space and* $f, g\colon X \to [-\infty, \infty]$ *are measurable functions. Then*

$$\max\{f(x), g(x)\} \qquad \text{and} \qquad \min\{f(x), g(x)\}$$

are measurable functions. Also,

$$f^+(x) = \max\{f(x), 0\} \qquad \text{and} \qquad f^-(x) = -\min\{f(x), 0\}$$

are measurable functions.

The above functions f^+ and f^- are called positive and negative parts of f, respectively. Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Theorem 3.14. *If $f = g - h$ and $g \geq 0$, $h \geq 0$, then $f^+ \leq g$ and $f^- \leq h$.*

Exercise 16. Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is a Borel function.

Exercise 17. Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a Borel function.

Exercise 18 (Bonus). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function, i.e. $f(x_1) \leq f(x_2)$ for $x_1 \leq x_2$. Show that f is a Borel function.

Definition 3.15. A function $f\colon X \to Y$ is a **simple function** iff its range $f(X)$ is finite.

If $s\colon X \to \mathbb{R}$ is a measurable simple function whose (distinct) values are $\alpha_1, \dots, \alpha_n$, then it can be represented by

$$s = \sum_{i=1}^{n} \alpha_i \, \chi_{A_i} \tag{3.1}$$

where $A_i = s^{-1}(\{\alpha_i\})$ are disjoint measurable sets such that $X = A_1 \cup \cdots \cup A_n$.

Theorem 3.16. *Let $(X, \mathfrak{M})$ be a measurable space and $f\colon X \to [0, \infty]$ a measurable function. Then there exist simple functions $s_n\colon X \to \mathbb{R}$ such that*

$$0 \leq s_1 \leq s_2 \leq \cdots \leq f$$

and $s_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$.

4 Lebesgue integration

DEFINITION 4.1. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $s\colon X \to [0, \infty)$ a nonnegative measurable simple function represented by (3.1). Then we define the **Lebesgue integral**

$$\int_X s\,d\mu = \sum_{i=1}^{n} \alpha_i\,\mu(A_i)$$

and for any $E \in \mathfrak{M}$ we define the **Lebesgue integral**

$$\int_E s\,d\mu = \sum_{i=1}^{n} \alpha_i\,\mu(A_i \cap E).$$

It is possible that for some i we have $\alpha_i = 0$ and $\mu(A_i \cap E) = \infty$, then we use the rule $0 \cdot \infty = 0$.

Note: $\int_E s\,d\mu \in [0, \infty]$ for any such s. If $\mu(E) = 0$, then $\int_E s\,d\mu = 0$.
If $s \equiv c \geq 0$ is a constant function, then $\int_E c\,d\mu = c\,\mu(E)$.
Example: $\int_X \chi_A\,d\mu = \mu(A)$. Note: $\int_E 0\,d\mu = 0$, even if $\mu(E) = \infty$.

DEFINITION 4.2. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $f\colon X \to [0, \infty]$ a nonnegative measurable function (possibly taking value $+\infty$). Then for any $E \in \mathfrak{M}$ we define the **Lebesgue integral**

$$\int_E f\,d\mu = \sup_{s \in L_f} \int_E s\,d\mu,$$

where

$$L_f = \big\{s\colon X \to [0, \infty) \text{ measurable simple}, \quad 0 \leq s \leq f\big\}.$$

Note that $L_f \neq \emptyset$ as it always contains the function $s \equiv 0$.

For simple functions, the above two definitions are equivalent.

Theorem 4.3 (Basic properties of the Lebesgue integral)**.** *Let $f, g\colon X \to [0, \infty]$ be measurable functions and $A, B, E \in \mathfrak{M}$ measurable sets. Then*

(a) *if $f \leq g$, then $\int_E f\,d\mu \leq \int_E g\,d\mu$;*

(b) *if* $A \subset B$, *then* $\int_A f\,d\mu \leq \int_B f\,d\mu$;

(c) *for any constant* $c \geq 0$ *we have* $\int_A cf\,d\mu = c\int_A f\,d\mu$;

(d) *if* $\mu(E) = 0$, *then* $\int_E f\,d\mu = 0$, *even if* $f \equiv \infty$;

(e) $\int_E f\,d\mu = \int_X \chi_E f\,d\mu$.

Exercise 19. Let $x_0 \in X$ and $\mu = \delta_{x_0}$ the δ-measure. Assume that $\{x_0\} \in \mathfrak{M}$. Show that for every measurable function $f\colon X \to [0,\infty]$ we have

$$\int_X f\,d\mu = f(x_0).$$

Exercise 20. Let $X = \mathbb{N}$ and μ the counting measure on the σ-algebra $\mathfrak{M} = 2^{\mathbb{N}}$. Show that for every function $f\colon X \to [0,\infty]$ we have

$$\int_X f\,d\mu = \sum_{n=1}^{\infty} f(n).$$

Theorem 4.4. *Let* $(X,\mathfrak{M},\mu)$ *be a measure space and* $s\colon X \to [0,\infty)$ *a nonnegative measurable simple function. Then*

$$\varphi(E) = \int_E s\,d\mu$$

is a measure on $\mathfrak{M}$.

Theorem 4.5. *Let* $(X,\mathfrak{M},\mu)$ *be a measure space and* $s,t\colon X \to [0,\infty)$ *two nonnegative measurable simple functions. Then*

$$\int_X (s+t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu.$$

Theorem 4.6 (Lebesgue's Monotone Convergence)**.** *Let* $(X,\mathfrak{M},\mu)$ *be a measure space and* $\{f_n\}$ *a sequence of measurable functions on* X. *Suppose that*
(a) $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq \infty$ *for every* $x \in X$;
(b) $f_n(x) \to f(x)$ *as* $n \to \infty$ *for every* $x \in X$.
Then f *is measurable and*

$$\int_X f_n\,d\mu \to \int_X f\,d\mu \qquad \text{as} \quad n \to \infty.$$

This can be written as

$$\lim_{n\to\infty}\int_X f_n\,d\mu = \int_X \lim_{n\to\infty} f_n\,d\mu.$$

EXERCISE 21. Let $f\colon X\to[0,\infty]$ and $\int_X f\,d\mu=0$. Show that $\mu\{x\colon f(x)\neq 0\}=0$.

Lemma. *Let $(X,\mathfrak{M},\mu)$ be a measure space and $f,g\colon X\to[0,\infty]$ two nonnegative measurable functions. Then*

$$\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu.$$

Theorem 4.7. *Let $(X,\mathfrak{M},\mu)$ be a measure space and $f_n\colon X\to[0,\infty]$ a sequence of nonnegative measurable functions on X and*

$$f(x)=\sum_{n=1}^{\infty} f_n(x)\quad\text{for every}\quad x\in X.$$

Then f is measurable and

$$\int_X f\,d\mu = \sum_{n=1}^{\infty}\int_X f_n\,d\mu.$$

That is, summation and integration "commute" for positive functions.

Corollary 4.8. *If $a_{ij}\geq 0$ for all $i,j=1,2,\ldots$, then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Theorem 4.9 (Fatou's Lemma)**.** *Let $(X,\mathfrak{M},\mu)$ be a measure space and $f_n\colon X\to[0,\infty]$ a sequence of nonnegative measurable functions on X. Then*

$$\int_X \left(\liminf_{n\to\infty} f_n\right)d\mu \leq \liminf_{n\to\infty}\int_X f_n\,d\mu.$$

Theorem 4.10. *Let $(X,\mathfrak{M},\mu)$ be a measure space and $f\colon X\to[0,\infty]$ a nonnegative measurable function on X. Then*

$$\varphi(E)=\int_E f\,d\mu$$

is a measure on $\mathfrak{M}$*. Furthermore,*

$$\int_X g\,d\varphi = \int_X gf\,d\mu$$

for every nonnegative measurable function g *on* X*.* [*We write* $d\varphi = f\,d\mu$*.*]

Corollary 4.11. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and* $f\colon X \to [0, \infty]$ *a nonnegative measurable function on* X *. Then for every* $A, B \in \mathfrak{M}$*,* $A \cap B = \emptyset$

$$\int_{A\cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu.$$

Exercise 22. Let $E \subset X$ be such that $\mu(E) > 0$ and $\mu(E^c) > 0$. Put $f_n = \chi_E$ if n is odd and $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

Exercise 23. Construct an example of a sequence of nonnegative measurable functions $f_n\colon X \to [0, \infty)$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists pointwise, but

$$\int_X f\,d\mu < \liminf_{n\to\infty} \int_X f_n\,d\mu.$$

Next we define Lebesgue integral for general complex-valued functions on a measure space $(X, \mathfrak{M}, \mu)$.

Definition 4.12. We say that a measurable function $f\colon X \to \mathbb{C}$ is **Lebesgue integrable** if

$$\int_X |f|\,d\mu < \infty.$$

The set of all integrable functions is denoted by $L^1(\mu)$.

Note: $|f|$ is measurable due to Theorem 3.9. The above integral is defined since $|f| \geq 0$.

Definition 4.13. Let $f \in L^1(\mu)$ and $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are its real and imaginary parts, so that $f = u + \mathbf{i}v$. Let $u = u^+ - u^-$ and $v = v^+ - v^-$ be the decomposition of u and v into their positive and negative parts. Then for any $E \in \mathfrak{M}$ we define the **Lebesgue integral** by

$$\int_E f\,d\mu = \int_E u^+\,d\mu - \int_E u^-\,d\mu + \mathbf{i}\int_E v^+\,d\mu - \mathbf{i}\int_E v^-\,d\mu.$$

Note: $u^{\pm}, v^{\pm} \geq 0$ and measurable by Corollary 3.13, so all the integrals are defined.

Note: $|u^{\pm}| \leq |u| \leq |f|$ and $|v^{\pm}| \leq |v| \leq |f|$, so all the integrals are finite, hence $\int_E f\,d\mu \in \mathbb{C}$.

For a real-valued function f we simply have $\int_E f\,d\mu = \int_E f^+\,d\mu - \int_E f^-\,d\mu$. This implies that

$$\operatorname{Re}\int_E f\,d\mu = \int_E \operatorname{Re} f\,d\mu \quad \text{and} \quad \operatorname{Im}\int_E f\,d\mu = \int_E \operatorname{Im} f\,d\mu.$$

Note: if $A \cap B = \emptyset$, then $\int_{A\cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu$.

Theorem 4.14. *Let $f, g \in L^1(\mu)$ and $\alpha, \beta \in \mathbb{C}$. Then*

$$\int_X (\alpha f + \beta g)\,d\mu = \alpha \int_X f\,d\mu + \beta \int_X g\,d\mu.$$

Exercise 24. Let $f, g \in L^1(\mu)$ be real-valued functions and $f \leq g$. Show that

$$\int_X f\,d\mu \leq \int_X g\,d\mu.$$

Exercise 25. Let $f_n \colon X \to [0,\infty]$ be a sequence of measurable functions such that $f_1 \geq f_2 \geq \cdots \geq 0$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. Suppose $f_1 \in L^1(\mu)$. Show that

$$\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X f\,d\mu.$$

Theorem 4.15. *If $f \in L^1(\mu)$, then*

$$\left|\int_X f\,d\mu\right| \leq \int_X |f|\,d\mu.$$

Theorem 4.16 (Lebesgue's Dominated Convergence)**.** *Let $f_n \colon X \to \mathbb{C}$ be a sequence of measurable functions such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in X$. Suppose there exists $g \in L^1(\mu)$ such that*

$$|f_n(x)| \leq g(x) \qquad \forall x \in X, \quad \forall n = 1, 2, \ldots$$

Then $f \in L^1(\mu)$,

$$\lim_{n\to\infty} \int_X |f_n - f|\,d\mu = 0$$

and

$$\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X f\,d\mu.$$

Corollary 4.17. *If $f \in L^1(\mu)$ and $E_1 \supset E_2 \supset \cdots$ is a sequence of measurable sets such that $\mu(E_n) \to 0$ as $n \to \infty$, then $\int_{E_n} f\,d\mu \to 0$ as $n \to \infty$.*

Next we discuss the role played by sets of measure zero (null sets).

DEFINITION 4.18. Let P be a property which a point $x \in X$ may or may not have. We say that P holds **almost everywhere (a.e.)** on a set $E \subset X$ if there exists $N \subset X$, $\mu(N) = 0$, such that P holds at every $x \in E \setminus N$.

Example: we say that $f = g$ a.e. if $\mu\{x \in X : f(x) \neq g(x)\} = 0$ (this is an equivalence relation). We say that a sequence of functions f_n converges a.e. to a limit function $f(x)$ if $\mu\{x \in X : f_n(x) \nrightarrow f(x)\} = 0$.

Theorem 4.19. *If $f = g$ a.e., then $\int_E f\,d\mu = \int_E g\,d\mu$ for any $E \in \mathfrak{M}$.*

DEFINITION 4.20. Let $(X, \mathfrak{M}, \mu)$ be a measure space and Y a topological space. We say that a function f with values in Y is **measurable and a.e. defined** on X if there exists $N \subset X$, $\mu(N) = 0$, such that f is defined on $X \setminus N$ and for every open set $V \subset Y$ we have $f^{-1}(V) \setminus N \in \mathfrak{M}$.

EXERCISE 26. Let f be a function as above. Fix a $y \in Y$ and define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \setminus N \\ y & \text{if } x \in N \end{cases}.$$

Show that $\tilde{f} : X \to Y$ is measurable.

EXERCISE 27. Let f be a function as above and μ a complete measure. Show that

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \setminus N \\ \text{any point of } Y & \text{if } x \in N \end{cases}.$$

Show that $\tilde{f} : X \to Y$ is measurable.

DEFINITION 4.21. A sequence f_n of functions $X \to \mathbb{C}$ converges to a function $f : X \to \mathbb{C}$ **uniformly on** X if for any $\varepsilon > 0$ there exists $N > 0$ such that for all $n > N$

$$\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon.$$

Equivalently, $\sup_{x \in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

EXERCISE 28. Suppose $\mu(X) < \infty$. Let a sequence $\{f_n\}$ of bounded complex measurable functions uniformly converge to f on X. Prove that $\lim_{n \to \infty} \int_X f_n\,d\mu = \int_X f\,d\mu$. Show that the assumption $\mu(X) < \infty$ cannot be omitted.

Theorem 4.22. *Let $\{f_n\}$ be a sequence of complex measurable functions defined a.e. on X and*

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty.$$

Then the series $f(x) = \sum_{n=1}^{\infty} f_n$ converges for a.e. $x \in X$ and

$$\sum_{n=1}^{\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Exercise 29. Give an example of a sequence of complex measurable functions $f_n \colon X \to \mathbb{C}$ (i.e., defined on the entire X) such that

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty$$

but the series $f(x) = \sum_{n=1}^{\infty} f_n$ diverges for some $x \in X$.

Definition 4.23. A sequence f_n of functions $X \to \mathbb{C}$ converges to a function $f \colon X \to \mathbb{C}$ **in measure** if for any $\varepsilon > 0$ and $\delta > 0$ there exists $N > 0$ such that for all $n > N$

$$\mu\{x \in X \colon |f_n(x) - f(x)| > \varepsilon\} < \delta.$$

Equivalently, $\mu\{x \in X \colon |f_n(x) - f(x)| > \varepsilon\} \to 0$ as $n \to \infty$.

Exercise 30. Suppose $\mu(X) < \infty$ and f_n are measurable functions defined a.e. on X. Prove that if $f_n \to f$ a.e. on X, then $f_n \to f$ in measure. What happens if $\mu(X) = \infty$?

Theorem 4.24. *Let $(X, \mathfrak{M}, \mu)$ be a measure space.*

(a) *If $f \colon X \to [0, \infty]$ is measurable, $E \in \mathfrak{M}$, and $\int_E f \, d\mu = 0$, then $f = 0$ a.e. on E;*

(b) *If $f \in L^1(\mu)$ and $\int_E f \, d\mu = 0$ for every $E \in \mathfrak{M}$, then $f = 0$ a.e. on X;*

(c) *If $f \in L^1(\mu)$ and*

$$\left| \int_X f \, d\mu \right| = \int_X |f| \, d\mu,$$

then there exists a constant $c \in \mathbb{C}$ such that $cf = |f|$ a.e. on X.

Theorem 4.25. *If $\mu(X) < \infty$, $f \in L^1(\mu)$, $S \subset \mathbb{C}$ closed, and for every $E \in \mathfrak{M}$ with $\mu(E) > 0$ we have $A_E(f) \in S$, where*

$$A_E(f) = \frac{1}{\mu(E)} \int_E f \, d\mu$$

is the average value of f on the set E, then $f(x) \in S$ a.e. on X.

Corollary 4.26. *In particular, if $\int_E f \, d\mu$ is real-valued for every E, then f is real-valued a.e. on X.*

Theorem 4.27. *If $E_k \in \mathfrak{M}$ and $\sum_{k=1}^{\infty} \mu(E_k) < \infty$, then almost every $x \in X$ belongs to finitely many of the sets E_k.*

Exercise 31. In the above theorem, let A be the set of points which belong to infinitely many of the sets E_k. Show that

$$A = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k.$$

Use this fact to prove the theorem without any reference to integration. Hint: recall that every measure μ is countably subbaditive, i.e. for any $E_n \in \mathfrak{M}$ (not necessarily disjoint) $\mu(\cup E_n) \le \sum \mu(E_n)$.

Exercise 32. Prove that if $f_n \to f$ in measure, then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ a.e. on X. Hint: use the previous theorem.

Exercise 33. Suppose $f \in L^1(\mu)$. Prove that $\forall \varepsilon > 0 \ \exists \delta > 0$ such that $\int_E |f| \, d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

5 Regular measures in $\mathbb{R}^k$

In Section 1 we constructed the Lebesgue measure on $\mathbb{R}^2$ (as well as on $\mathbb{R}$, and our construction easily extends to $\mathbb{R}^k$ for $k \ge 3$). Here we continue the study of the Lebesgue measure on $\mathbb{R}^k$.

Definition 5.1. A measure μ defined on a σ-algebra $\mathfrak{M}$ in $\mathbb{R}^k$ is called a Borel measure if it is defined on all Borel sets (i.e. $\mathfrak{M}$ contains all Borel sets).

A measure μ is Borel if $\mathfrak{M}$ contains all open intervals.

Definition 5.2. A Borel measure μ is said to be **outer regular** if

$$\mu(E) = \inf\{\mu(V) \colon E \subset V, \ \ V \text{ open}\} \quad \forall E \in \mathfrak{M}.$$

A Borel measure μ is said to be **inner regular** if

$$\mu(E) = \sup\{\mu(K)\colon K \subset E, \quad K \text{ compact}\} \quad \forall E \in \mathfrak{M}.$$

A Borel measure μ is said to be **regular** if it is both outer regular and inner regular.

Theorem 5.3. *If $\mu(E) < \infty$ for any bounded set $E \subset \mathbb{R}^k$, then the outer regularity and inner regularity are equivalent.*

EXERCISE 34. Show that the counting measure on $\mathbb{R}^k$ is inner regular, but not outer regular.

Theorem 5.4. *The Lebesgue measure on $\mathbb{R}^k$ is regular.*

EXERCISE 35. Find examples of Lebesgue measurable sets $E_1, E_2 \subset \mathbb{R}$ such that

$$\mathbf{m}(E_1) < \inf\{\mathbf{m}(A)\colon E \subset A, \quad A \text{ closed}\}$$

$$\mathbf{m}(E_2) > \sup\{\mathbf{m}(V)\colon V \subset E, \quad V \text{ open}\}.$$

where $\mathbf{m}$ is the Lebesgue measure.

Theorem 5.5. *Let μ be a regular Borel measure. Then for every $E \in \mathfrak{M}$*

(a) *there exists a G_δ-set G such that $E \subset G$ and $\mu(E) = \mu(G)$;*

(b) *there exists an F_σ-set F such that $F \subset E$ and $\mu(E) = \mu(F)$.*

So every measurable set is 'almost' G_δ (and 'almost' F_σ), up to a null set.

Theorem 5.6. *There exists a unique measure μ defined on the Borel σ-algebra in $\mathbb{R}$ such that $\mu(I) = |I|$ (the length of I) for every open interval $I \subset \mathbb{R}$.*

This theorem can be extended to $\mathbb{R}^2$ as follows: there exists a unique measure μ defined on the Borel σ-algebra in $\mathbb{R}^2$ such that $\mu(R) = m(R)$ (the area of R) for every open rectangle $R \subset \mathbb{R}^2$. But the proof becomes more complicated.

Next recall that the Lebesgue measure $\mathbf{m}$ on $\mathbb{R}^2$ is complete.

Theorem 5.7. *If μ is another complete measure on $\mathbb{R}^2$ defined (at least) on all Borel sets such that $\mu(R) = \text{Area}(R)$ for every rectangle R, then $\mu(E) = \mathbf{m}(E)$ for every Lebesgue measurable set E.*

Recall that a measure μ on $\mathbb{R}$ (or $\mathbb{R}^2$) is called translation invariant if $\mu(E) = \mu(E+a)$ for every measurable set E and $a \in \mathbb{R}$ (respectively, $a \in \mathbb{R}^2$).

Theorem 5.8. *Let μ be a translation invariant Borel measure on $\mathbb{R}$ such that $\mu(I) < \infty$ for at least one open interval $I \neq \emptyset$. Then there exists a constant c such that $\mu(E) = c\mathbf{m}(E)$ for every Borel set E; here $\mathbf{m}$ is the Lebesgue measure.*

Exercise 36. Extend this theorem to $\mathbb{R}^2$: show that if μ is a translation invariant Borel measure on $\mathbb{R}^2$ such that $\mu(R) < \infty$ for at least one open rectangle $R \neq \emptyset$, then there exists a constant c such that $\mu(E) = c\mathbf{m}(E)$ for every Borel set E.

Exercise 37. Let $f\colon \mathbb{R} \to [0, \infty]$ be a Borel function, $f \in L^1(\mathbf{m})$, and consider the measure $\rho(E) = \int_E f\, d\mathbf{m}$, where $\mathbf{m}$ is the Lebesgue measure. Prove that ρ is regular. Hint: use the result of Exercise 33.

6 Approximation of Lebesgue measurable functions

In the previous section we have seen that Lebesgue measurable sets can be arbitrarily well approximated by open sets ('from outside') and by compact sets ('from inside'). Here we approximate Lebesgue measurable functions by step functions and by continuous functions.

Definition 6.1. A function $\varphi\colon \mathbb{R} \to \mathbb{C}$ is called a **step function** if $\varphi = \sum_{i=1}^n \alpha_i \chi_{I_i}$ for some $\alpha_i \in \mathbb{C}$ and disjoint bounded intervals $I_i \in \mathbb{R}$.

Exercise 38. Let $s\colon [a, b] \to \mathbb{R}$ be a simple Lebesgue measurable function. Show that for every $\varepsilon > 0$ there is a step function $\varphi\colon [a, b] \to \mathbb{R}$ and a Lebesgue measurable set $E \subset [a, b]$ such that $s(x) = \varphi(x)$ on E and $\mu\big([a, b] \setminus E\big) < \varepsilon$. Hint: use the regularity of μ.

Exercise 39. Let $f\colon [a, b] \to \mathbb{R}$ be a Lebesgue measurable function. Show that for every $\varepsilon > 0$ there is a step function $g\colon [a, b] \to \mathbb{R}$ such that

$$\mu\big\{x \in [a, b] \colon |f(x) - g(x)| \geq \varepsilon\big\} < \varepsilon.$$

Hint: first show that $|f| \leq M$ except for a set of small measure, then use the previous exercise.

Exercise 40. Let $f \in L^1$ with respect to the Lebesgue measure $\mathbf{m}$ on $\mathbb{R}$. Prove that there is a sequence $\{g_n\}$ of step functions such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f - g_n|\, d\mathbf{m} = 0.$$

Hint: use the previous exercise.

Definition 6.2. Let X be a topological space and $f\colon X \to \mathbb{C}$. The **support** of f is defined by
$$\operatorname{supp} f = \overline{\{x \in X \colon f(x) \neq 0\}}.$$
Also, $C_c(X)$ denotes the set of all continuous functions with compact support.

Theorem 6.3 (Lusin). *Let $f\colon \mathbb{R} \to \mathbb{C}$ be a Lebesgue measurable function, $A \subset \mathbb{R}$ a Lebesgue measurable set with $\mu(A) < \infty$ (here μ is the Lebesgue measure), such that $f(x) = 0$ for all $x \notin A$. Then for every $\varepsilon > 0$ there exists $g \in C_c(\mathbb{R})$ such that*
$$\mu\big\{x \colon f(x) \neq g(x)\big\} < \varepsilon.$$
Furthermore, we may arrange it so that $\sup|g(x)| \leq \sup|f(x)|$.

Corollary 6.4. *Assume that the hypotheses of Lusin's theorem are satisfied and that $|f| \leq M$. Then there is a sequence $\{g_n\}$ such that $g_n \in C_c(\mathbb{R})$, $|g_n| \leq M$, and $g_n(x) \to f(x)$ a.e.*

Definition 6.5. Let X be a topological space and $f\colon X \to \mathbb{R}$ (or $[-\infty, \infty]$). We say that f is **lower semicontinuous** if $\{x \colon f(x) > a\}$ is open for every $a \in \mathbb{R}$. We say that f is **upper semicontinuous** if $\{x \colon f(x) < a\}$ is open for every $a \in \mathbb{R}$.

Proposition 6.6. *Simple properties of semicontinuous functions:*

(a) *f is continuous $\Leftrightarrow$ f is both upper and lower semicontinuous;*

(b) *if f is semicontinuous, then it is Borel;*

(c) *f is upper semicontinuous $\Leftrightarrow$ $-f$ is lower semicontinuous;*

(d) *if f is upper (lower) semicontinuous and $c > 0$, then cf is upper (lower) semicontinuous;*

(e) *if f, g are upper (lower) semicontinuous, then $f + g$ is upper (lower) semicontinuous;*

(f) *if $\{f_\gamma\}$ is a family of upper (lower) semicontinuous functions, then $\inf f_\gamma$ (resp., $\sup f_\gamma$)is upper (lower) semicontinuous;*

(g) *$V \subset X$ is open $\Leftrightarrow$ χ_V is lower semicontinuous;*

(h) *$A \subset X$ is closed $\Leftrightarrow$ χ_A is upper semicontinuous;*

(i) *if* $f_n \geq 0$ *are lower semicontinuous, then* $\sum f_n$ *is lower semicontinuous.*

EXERCISE 41. Prove that if $f\colon X \to \mathbb{R}$ is upper (lower) semicontinuous and X is compact, then f is bounded above (below) and attains its maximum (minimum).

Theorem 6.7 (Vitali-Caratheodory)**.** *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be Lebesgue integrable function (i.e.* $f \in L^1$*). Then for every* $\varepsilon > 0$ *there exist functions* $u, v\colon \mathbb{R} \to \mathbb{R}$, $u \leq f \leq v$, *where* u *is upper semicontinuous and bounded above,* v *is lower semicontinuous and bounded below, and*

$$\int_{\mathbb{R}} (v - u)\, d\mu < \varepsilon.$$

Theorem 6.8. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be Lebesgue measurable function. Then there exist Borel functions* $g, h\colon \mathbb{R} \to \mathbb{R}$ *such that* $g(x) = h(x)$ *a.e. and* $g(x) \leq f(x) \leq h(x)$ *for every* $x \in X$.

Theorem 6.9 (Egorov)**.** *Let* $\mu(X) < \infty$ *and* $\{f_n\}$ *a sequence of complex measurable functions on* X *such that* $\lim_{n\to\infty} f_n(x) = f(x)$ *for every* $x \in X$. *Then for every* $\varepsilon > 0$ *there is* $E \subset X$ *such that* $\mu(X \setminus E) < \varepsilon$ *and* $f_n \to f$ *uniformly on* E.

7 Lebesgue integral versus Riemannian integral

Theorem 7.1. *Let* $[a, b] \subset \mathbb{R}$ *be bounded. If the Riemannian integral*

$$I = \int_a^b f(x)\, dx$$

exists, then f *is Lebesgue integrable on* $[a, b]$ *and its Lebesgue integral equals*

$$\int_{[a,b]} f\, d\mathbf{m} = I.$$

Theorem 7.2. *The Riemannian integral*

$$\int_a^b f(x)\, dx$$

exists if and only if f *is bounded and almost everywhere continuous.*

Note: the existence of improper Riemannian integrals do not imply Lebesgue integrability. More precisely, if f has a finite improper Riemannian integral, then it is Lebesgue integrable if and only if $|f|$ has a finite improper Riemannian integral. In that case the Lebesgue integral is equal to the improper Riemannian integral. (Note: if $f \geq 0$, then the existence of improper Riemannian integral is equivalent to Lebesgue integrability.)

EXERCISE 42. Find a function $f(x)$ on $[0, \infty)$ such that the improper Riemannian integral $\int_0^\infty f(x)\,dx = \lim_{A\to\infty} \int_0^A f(x)\,dx$ exists (is finite), but f is not Lebesgue integrable.

8 L^p spaces

DEFINITION 8.1. A function $\varphi\colon (a,b) \to \mathbb{R}$, where $-\infty \leq a < b \leq \infty$ is said to be **convex** if

$$\varphi(px + qy) \leq p\varphi(x) + q\varphi(y)$$

for all $a < x \leq y < b$ and $p, q > 0$, $p + q = 1$.

Lemma 8.2. *A function $f\colon (a,b) \to \mathbb{R}$ is convex if and only if for any $a < s < t < u < b$*

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}.$$

Lemma 8.3. *If φ is differentiable, then it is convex if and only if φ' is monotonically increasing, i.e. $\varphi'(x) \leq \varphi'(y)$ for all $x < y$.*

Theorem 8.4. *If φ is convex on (a,b), then it is continuous on (a,b).*

Note: this is not true on closed intervals $[a,b]$.

Theorem 8.5 (Jensen's inequality)**.** *Let $(\Omega, \mathfrak{M}, \mu)$ be a measure space and $\mu(\Omega) = 1$. If $f\colon \Omega \to (a,b) \subset \mathbb{R}$ is Lebesgue integrable ($f \in L^1_\mu$) and φ is a convex function on (a,b), then*

$$\varphi\Big(\int_\Omega f\,d\mu\Big) \leq \int_\Omega (\varphi \circ f)\,d\mu.$$

Note: the cases $a = -\infty$ and $b = \infty$ are not excluded.

EXERCISE 43. Prove that the supremum of any collection of convex functions on (a,b) is convex on (a,b) (assume that the supremum is finite). Prove that a pointwise limit of a sequence of convex functions is convex.

EXERCISE 44. Let φ be convex on (a, b) and ψ convex and nondecreasing on the range of φ. Prove that $\psi \circ \varphi$ is convex on (a, b). For $\varphi > 0$, show that the convexity of $\log \varphi$ implies the convexity of φ, but not vice versa.

EXERCISE 45. Assume that φ is a continuous real function on (a, b) such that

$$\varphi\Big(\frac{x+y}{2}\Big) \leq \frac{1}{2}\,\varphi(x) + \frac{1}{2}\,\varphi(y)$$

for all $x, y \in (a, b)$. Prove that φ is convex. (The conclusion does not follow if continuity is omitted from the hypotheses.)

Example: let $\Omega = \{1, 2, \ldots, n\}$ and $\mu(\{i\}) = 1/n$ for every $i = 1, \ldots, n$; let $f(i) = \ln a_i$ for some $a_1, \ldots, a_n > 0$ and $\varphi(x) = e^x$ (this is a convex function!). Then Jensen's inequality implies

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

(geometric mean is $\leq$ arithmetic mean). More general: if $\mu(\{i\}) = p_i$ and $p_1 + \cdots + p_n = 1$, then

$$a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \leq p_1 a_1 + p_2 a_2 + \cdots + p_n a_n$$

DEFINITION 8.6. If $p, q > 0$ and

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then we say that p and q are **conjugate exponents**.

Note: we must have $p, q > 1$. If $p \to 1$, then $q \to \infty$; thus $p = 1$ and $q = \infty$ are sometimes considered as conjugate exponents, too.

An important special case is $p = q = 2$.

Theorem 8.7. *Let p and q be conjugate exponents, $(X, \mathfrak{M}, \mu)$ a measure space, and $f, g\colon X \to [0, \infty]$ measurable functions. Then*

$$\int_X fg\, d\mu \leq \left[\int_X f^p\, d\mu\right]^{1/p} \left[\int_X g^q\, d\mu\right]^{1/q}$$

(**Hölder inequality**, in the case $p = q = 2$ it is **Schwarz inequality**) *and*

$$\left[\int_X (f+g)^p\, d\mu\right]^{1/p} \leq \left[\int_X f^p\, d\mu\right]^{1/p} + \left[\int_X g^p\, d\mu\right]^{1/p}$$

(**Minkowski inequality**).

Remark 8.8. *Equality in the Hölder inequality holds if and only if there exist $\alpha, \beta \geq 0$, not both equal to 0, such that $\alpha f^p = \beta g^q$ a.e.*

EXERCISE 46. Prove that equality in the Minkowski inequality holds if and only if there exist $\alpha, \beta \geq 0$, not both equal to 0, such that $\alpha f = \beta g$ a.e.

EXERCISE 47. Suppose $\mu(\Omega) = 1$ and suppose f and g are two positive measurable functions on Ω such that $fg \geq 1$. Prove that

$$\int_\Omega f \, d\mu \cdot \int_\Omega g \, d\mu \geq 1.$$

EXERCISE 48. Suppose $\mu(\Omega) = 1$ and $h \colon \Omega \to [0, \infty]$ is measurable. Denote $A = \int_\Omega h \, d\mu$. Prove that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2} \, d\mu \leq 1 + A.$$

EXERCISE 49 (Bonus). If $\mathbf{m}$ is Lebesgue measure on $[0, 1]$ and if h is a continuous function on $[0, 1]$ such that $h = f'$, then the inequalities in the previous exercise have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

Recall that $L^1_\mu(X) = \{f \colon \int_X |f| \, d\mu < \infty\}$.

DEFINITION 8.9. For a measurable function $f \colon X \to \mathbb{C}$ and $p > 0$ we define its **"p-norm"** by

$$\|f\|_p = \left[\int_X |f|^p \, d\mu \right]^{1/p}.$$

We also define the space of p-integrable functions by

$$L^p_\mu(X) = \left\{ f \colon \|f\|_p < \infty \right\}$$

If μ is the Lebesgue measure on $\mathbb{R}^k$, we will simply write $L^p(\mathbb{R}^k)$.

DEFINITION 8.10. A special case: μ is a counting measure on a countable set A. Then we identify A with $\mathbb{N}$ and functions $f \colon A \to \mathbb{C}$ with complex sequences $\xi = \{\xi_n\}$. Then

$$\|\xi\|_p = \left[\sum_{n=1}^\infty |\xi_n|^p \right]^{1/p}$$

and

$$\ell^p = \left\{ \xi \colon \|\xi\|_p < \infty \right\}$$

Proposition 8.11. *If $1 \le p < \infty$, then L^p_μ is a complex vector space.*

Next we investigate the case $p = \infty$. Let $g\colon X \to [0, \infty]$ be measurable. The set

$$S = \big\{\alpha \in \mathbb{R} \colon \mu\big(g^{-1}(\alpha, \infty]\big) = 0\big\}$$

is either empty or a closed interval $[\beta, \infty)$. If $S = \emptyset$ we set $\beta = \infty$.

Definition 8.12. The value of β is called the **essential supremum** and denoted by

$$\text{ess-sup}\, g = \beta.$$

The **infinity-norm** of a function $f\colon X \to \mathbb{C}$ is defined by

$$\|f\|_\infty = \text{ ess-sup}\,|f|.$$

We also denote by

$$L^\infty_\mu(X) = \big\{f \colon \|f\|_\infty < \infty\big\}$$

the space of **essentially bounded functions**.

Note: $|f(x)| \le \|f\|_\infty$ a.e. A special case: ℓ^∞ is the space of bounded sequences.

Proposition 8.13. *L^∞_μ is a complex vector space.*

Theorem 8.14. *Let p and q be complex conjugate exponents, including the limiting cases $p = 1$, $q = \infty$ and $p = \infty$, $q = 1$. Then for every $f \in L^p$ and $g \in L^q$ we have $fg \in L^1$ and*

$$\|fq\|_1 \le \|f\|_p\, \|g\|_q.$$

Exercise 50. When does one get equality in $\|fg\|_1 \le \|f\|_\infty\, \|g\|_1$?

Theorem 8.15. *Let $1 \le p \le \infty$. Then for every $f, g \in L^p$ we have $f + g \in L^q$ and*

$$\|f + q\|_p \le \|f\|_p + \|g\|_p.$$

We also have

$$\|cf\|_p = |c|\, \|f\|_p.$$

Nonetheless, $\|\cdot\|_p$ is not a norm as there are nonzero functions f with $\|f\|_p = 0$. But we will 'reduce' the L^p space so that $\|\cdot\|_p$ will be a true norm.

Let V be a complex vector space and $W \subset V$ a subspace. We say that $v_1 \sim v_2$ iff $v_1 - v_2 \in W$. This is an equivalence relation. Denote by

$$[v] = \{v_1 \in V \colon v - v_1 \in W\}$$

the equivalence class containing v and

$$V/W = \{[v] \colon v \in V\}$$

the set of equivalence classes. Then V/W is a vector space with

$$[v] + [w] = [v + w], \qquad c[v] = [cv].$$

We call V/W the quotient space.

Definition 8.16. Let $(X, \mathfrak{M}, \mu)$ be a measure space and

$$\mathcal{N}_\mu = \{f \colon X \to \mathbb{C} \text{ measurable}, \quad f = 0 \text{ a.e.}\}$$

For every $1 \leq p \leq \infty$ define

$$\mathcal{L}^p_\mu = L^p_\mu / \mathcal{N}_\mu.$$

Note: $\mathcal{L}^p_\mu$ is obtained from L^p_μ by identifying functions that coincide a.e.

Theorem 8.17. *Let $1 \leq p \leq \infty$. Then $\mathcal{L}^p_\mu$ is a vector space with norm $\|\cdot\|_p$.*

Exercise 51. Suppose $f \colon X \to \mathbb{C}$ is measurable and $\|f\|_\infty > 0$. Define

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \qquad (0 < p < \infty)$$

and consider the set $E = \{p \colon \varphi(p) < \infty\}$. Each of the following questions is graded as a separate exercise. Question (c) and (e) are **bonus** problems.

(a) Let $r < p < s$ and $r, s \in E$. Prove that $p \in E$.

(b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \leq \max(\|f\|_r, \|f\|_s)$. Show that this implies the inclusion $L^r_\mu \cap L^s_\mu \subset L^p_\mu$.

(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

Recall that if V is a vector space with a norm $\|\cdot\|$, we can define distance (metric) on V by

$$d(u, v) = \|u - v\|.$$

Thus the space $\mathcal{L}^p_\mu$ with norm $\|\cdot\|_p$ becomes a metric space.

DEFINITION 8.18. A metric is **complete** if every Cauchy sequence converges to a limit. A vector space V with norm $\|\cdot\|$ that induces a complete metric on it is called **Banach space**.

Theorem 8.19. *Let $1 \le p \le \infty$. Then $\mathcal{L}^p_\mu$ is a Banach space.*

Corollary 8.20. *Let $1 \le p \le \infty$. If $f_n \to f$ in the L^p_μ metric, then there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e.*

EXERCISE 52. Let $\mu(X) = 1$. Each of the following questions is graded as a separate exercise.

(a) Prove that $\|f\|_r \le \|f\|_s$ if $0 < r < s \le \infty$.

(b) Under what condition does it happen that $0 < r < s \le \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(c) Prove that $L^r_\mu \supset L^s_\mu$ if $0 < r < s$. If $X = [0, 1]$ and $\mathbf{m}$ is the Lebesgue measure, show that $L^r_{\mathbf{m}} \ne L^s_{\mathbf{m}}$.

EXERCISE 53 (Bonus). For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ if $r \ne s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

EXERCISE 54. (a) Show that $\int_0^{\frac{\pi}{2}} \sqrt{x \sin x}\, dx < \frac{\pi}{2\sqrt{2}}$;
(b) Show that $\left[\int_0^1 x^{\frac{1}{2}}(1-x)^{-\frac{1}{3}}\, dx\right]^3 \le \frac{8}{5}$.

Next we approximate L^p functions by simple functions and continuous functions. Let

$$S = \Big\{s \colon X \to \mathbb{C} \ \text{ simple, measurable,} \quad \mu(s \ne 0) < \infty\Big\}.$$

Theorem 8.21. *S is dense in L^p for every $1 \le p < \infty$.*

Note: this is not true for $p = \infty$ (for example, $X = \mathbb{R}$ and $f \equiv 1$).

Recall that $C_c(\mathbb{R})$ denotes the space of continuous functions $f \colon \mathbb{R} \to \mathbb{C}$ with compact support. And again, in $\mathbb{R}$ we use the Lebesgue measure $\mathbf{m}$.

Theorem 8.22. *C_c is dense in L^p_μ for every $1 \le p < \infty$.*

Note: this is not true for $p = \infty$ (for example, $X = \mathbb{R}$ and $f \equiv 1$).

Remark 8.23. *Step functions $f\colon \mathbb{R} \to \mathbb{C}$ are dense in L^p.*

We can say that the L^p space, for $1 < p < \infty$, is the completion (in the p-metric) of the space of continuous functions. This is not true for $p = \infty$.

DEFINITION 8.24. A function $f\colon \mathbb{R} \to \mathbb{C}$ is said to **vanish at infinity** if $f(x) \to 0$ as $|x| \to \infty$. The space of continuous functions vanishing at infinity is denoted by $C_0(\mathbb{R})$.

Note that $C_c(\mathbb{R}) \subset C_0(\mathbb{R})$.

Theorem 8.25. *The completion of $C_c(\mathbb{R})$ in the $\|\cdot\|_\infty$-metric is $C_0(\mathbb{R})$ (modulo the identification of equivalent functions).*

Corollary 8.26. *$C_0(\mathbb{R})$ with norm $\|\cdot\|_\infty$ is a Banach space.*

EXERCISE 55 (Bonus). Suppose $1 < p < \infty$ and $f \in L^p((0,\infty))$ relative to the Lebesgue measure. Define

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt \qquad (0 < x < \infty).$$

Prove Hardy's inequality

$$\|F\|_p \le \frac{p}{p-1}\|f\|_p$$

which shows that the mapping $f \to F$ carries L^p into L^p. [Hint: assume first that $f \ge 0$ and $f \in C_c((0,\infty))$, i.e. the support of f is a finite closed interval $[a,b] \subset (0,\infty)$. Then use integration by parts:

$$\int_\varepsilon^A F^p(x)\,dx = -p\int_\varepsilon^A F^{p-1}xF'(x)\,dx$$

where $\varepsilon < a$ and $A > b$. Note that $xF' = f - F$, and apply Hölder inequality to $\int F^{p-1}f\,dx$.]

9 Complex measures

Given two measures μ_1 and μ_2 on $(X, \mathfrak{M})$, we can define a new measure $\mu = \mu_1 + \mu_2$ by $\mu(A) = \mu_1(A) + \mu_2(A)$ for all $A \in \mathfrak{M}$ (it is easy to check that μ is a measure). Also for any measure μ_1 on $(X, \mathfrak{M})$ and a scalar $c \ge 0$ we can define a measure $\mu = c\mu_1$ by $\mu(A) = c\mu_1(A)$ for all $A \in \mathfrak{M}$. Thus the collection of measures on $(X, \mathfrak{M})$ is 'almost' a vector space, except we cannot subtract measures or multiply them by negative scalars.

Also recall that given a measure μ on $(X, \mathfrak{M})$ and a measurable function $f \geq 0$ on X we can define a measure λ by $\lambda(A) = \int_A f \, d\mu$ for all $A \in \mathfrak{M}$. (Recall that we write $d\lambda = f \, d\mu$ and call f the density of λ.) But what if f takes negative values? Or what if f takes complex values?

Definition 9.1. A function $\mu \colon \mathfrak{M} \to \mathbb{C}$ is called a **complex measure** if it is σ-additive, i.e. if for any countable collection of pairwise disjoint measurable sets $\{E_i\}_{i=1}^{\infty}$ we have

$$\mu\big(\cup_{i=1}^{\infty} E_i\big) = \sum_{i=1}^{\infty} \mu(E_i).$$

Note: we say that $\{E_i\}_{i=1}^{\infty}$ make a **partition** of $E = \cup_{i=1}^{\infty} E_i$ if E_i's are pairwise disjoint.

As opposed to complex measures, we will call earlier defined measures **positive measures**. Observe that not every positive measure is a complex measure (because positive measures can take infinite values).

Lemma 9.2. *The series $\sum_{i=1}^{\infty} \mu(E_i)$ in the above definition must converge absolutely.*

Definition 9.3. Let μ be a complex measure on $(X, \mathfrak{M})$. Define the function $|\mu|$ on $\mathfrak{M}$ by

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|.$$

where the supremum is taken over all measurable partitions of E. Then $|\mu|$ is called the **total variation** of μ.

Theorem 9.4. *Let μ be a complex measure on $(X, \mathfrak{M})$.*

(a) *The total variation $|\mu|$ is a positive measure on $(X, \mathfrak{M})$. It satisfies $|\mu(E)| \leq |\mu|(E)$ for every $E \in \mathfrak{M}$;*

(b) *If λ is a positive measure on $\mathfrak{M}$ with $|\mu(E)| \leq \lambda(E)$ for all $E \in \mathfrak{M}$, then $|\mu|(E) \leq \lambda(E)$ for all $E \in \mathfrak{M}$.*

Remark: the property (b) means that $|\mu|$ is the smallest positive measure that 'dominates' μ (in the sense of (b)).

Lemma 9.5. *If $z_1, \dots, z_N \in \mathbb{C}$, then there is a subset $S \subset \{1, \dots, N\}$ such that*
$$\left|\sum_{k\in S} z_k\right| \geq \frac{1}{\pi}\sum_{k=1}^{N} |z_k|.$$

Theorem 9.6. *If μ be a complex measure on $(X, \mathfrak{M})$, then $|\mu|$ is a finite measure, i.e. $|\mu|(X) < \infty$.*

Observe that the range of μ is bounded, since $|\mu(E)| \leq |\mu|(E) \leq |\mu|(X)$; i.e. all the values of μ lie in a closed disk D of radius $R = |\mu|(X)$, i.e. $\mu(E) \in D$ for all $E \in \mathfrak{M}$.

Exercise 56. Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^1_\mu$ define
$$\mu_f(E) = \int_E f\, d\mu.$$
Show:
(a) μ_f is a complex measure;
(b) $|\mu_f| = \mu_{|f|}$, assuming that f is real-valued;
(c) $|\mu_f| = \mu_{|f|}$, now for a general $f \in L^1(\mu)$.

We denote by $\mathbb{M}(X, \mathfrak{M})$ the set of all complex measures on $(X, \mathfrak{M})$. For any two measures $\mu_1, \mu_2 \in \mathbb{M}(X, \mathfrak{M})$ we can define a new measure $\mu = \mu_1 + \mu_2$ by $\mu(A) = \mu_1(A) + \mu_2(A)$ for all $A \in \mathfrak{M}$ (it is easy to check that μ is a measure). Also for any measure $\mu_1 \in \mathbb{M}(X, \mathfrak{M})$ on $(X, \mathfrak{M})$ and a scalar $c \in \mathbb{C}$ we can define a measure $\mu = c\mu_1$ by $\mu(A) = c\mu_1(A)$ for all $A \in \mathfrak{M}$.

Theorem 9.7. *$\mathbb{M}(X, \mathfrak{M})$ is a vector space with a norm $\|\mu\| = |\mu|(X)$.*

Exercise 57 (Bonus). Prove that the space $\mathbb{M}(X, \mathfrak{M})$ with the norm $\|\mu\|$ is a Banach space, i.e. it is a complete metric space (every Cauchy sequence converges to a limit). Hint: given a Cauchy sequence of complex measures $\{\mu_n\}$ you need to construct the limit measure μ and prove that $\|\mu_n - \mu\| \to 0$ as $n \to \infty$.

Definition 9.8. Let μ be a real measure on $(X, \mathfrak{M})$ (i.e. a real-valued complex measure). Define
$$\mu^+ = \tfrac{1}{2}(|\mu| + \mu) \qquad \text{and} \qquad \mu^- = \tfrac{1}{2}(|\mu| - \mu)$$
These are called **positive and negative variations** of μ.

Note that μ^+ and μ^- are positive measures that satisfy

$$\mu = \mu^+ - \mu^- \qquad \text{and} \qquad |\mu| = \mu^+ + \mu^-.$$

The first formula is called **Jordan decomposition** of μ. We will prove that μ^+ and μ^- are *minimal* measures that satisfy Jordan decomposition.

DEFINITION 9.9. Let μ be a positive measure and λ a complex measure on $(X, \mathfrak{M})$.

(a) we say that λ is **absolutely continuous** with respect to μ and write $\lambda \ll \mu$ if

$$\forall E \in \mathfrak{M}\colon\ \mu(E) = 0 \ \Rightarrow \lambda(E) = 0;$$

(b) we say that λ is **concentrated on** $A \in \mathfrak{M}$ if

$$\forall E \in \mathfrak{M}\colon\ \lambda(E) = \lambda(E \cap A);$$

(c) we say that two complex measures λ_1 and λ_2 are **mutually singular** and write $\lambda_1 \perp \lambda_2$ if λ_1 is concentrated on A_1 and λ_2 is concentrated on A_2 such that $A_1 \cap A_2 = \emptyset$.

Note: λ is concentrated on A if and only if

$$\forall E \in \mathfrak{M}\colon\ A \cap E = \emptyset \ \Rightarrow \lambda(E) = 0.$$

Note also that if $\lambda \ll \mu$ and μ is concentrated on A, then λ is concentrated on A as well. Suppose that $0 \neq c \in \mathbb{C}$; then $\lambda \ll \mu \iff c\lambda \ll \mu$; also λ is concentrated on A iff $c\lambda$ is concentrated on A.

Proposition 9.10. *Let μ be a positive measure and λ's complex measures on $(X, \mathfrak{M})$.*

(a) *if λ is concentrated on A, then $|\lambda|$ is concentrated on A;*

(b) *if $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$;*

(c) *if $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$;*

(d) *if $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$;*

(e) *if $\lambda \ll \mu$, then $|\lambda| \ll \mu$;*

(f) *if* $\lambda_1 \ll \mu$ *and* $\lambda_2 \perp \mu$*, then* $\lambda_1 \perp \lambda_2$*;*

(g) *if* $\lambda \ll \mu$ *and* $\lambda \perp \mu$*, then* $\lambda = 0$*;*

Example: let μ be a positive measure on $(X, \mathfrak{M})$ and $f \in L^1_\mu$. Then the measure μ_f defined by

$$\mu_f(E) = \int_E f \, d\mu$$

is absolutely continuous with respect to μ, i.e. $\mu_f \ll \mu$.

EXERCISE 58. Let μ be a positive measure on $(X, \mathfrak{M})$ and $f, g \in L^1_\mu$. Define μ_f and μ_g as above. Prove that

(a) μ_f is concentrated on A if and only if $\mu\{x \in A^c \colon f(x) \neq 0\} = 0$;

(b) $\mu_f \perp \mu_g$ if and only if $\mu\{x \in X \colon f(x)g(x) \neq 0\} = 0$;

(c) if $f \geq 0$, then

$$\mu \ll \mu_f \iff f(x) > 0 \text{ for } \mu - \text{a.e. } x \in X.$$

EXERCISE 59. Let λ be a positive measure on $(X, \mathfrak{M})$. Prove that λ is concentrated on A if and only if $\lambda(A^c) = 0$. Give a counterexample to this statement in the case of a complex measure λ.

DEFINITION 9.11. A positive measure μ on $(X, \mathfrak{M})$ is said to be σ-**finite** if there exist a countable sequence of sets $E_n \in \mathfrak{M}$ such that $X = \cup_n E_n$ and $\mu(E_n) < \infty$ for all n.

Theorem 9.12 (Lebesgue-Radon-Nikodym)**.** *Let* μ *be a positive* σ*-finite measure and* λ *a complex measure on* $(X, \mathfrak{M})$*.*

(a) *There exist a unique pair of complex measures* λ_{a} *and* λ_{s} *such that*

$$\lambda = \lambda_{\mathrm{a}} + \lambda_{\mathrm{s}}$$

and $\lambda_{\mathrm{a}} \ll \mu$ *and* $\lambda_{\mathrm{s}} \perp \mu$*;*

(b) *There is a unique* $h \in L^1_\mu$ *(up to* μ*-equivalence) such that*

$$\lambda_{\mathrm{a}}(E) = \int_E h \, d\mu$$

i.e. $\lambda_{\mathrm{a}} = \mu_h$*.*

Note that $\lambda_{\mathrm{a}} \perp \lambda_{\mathrm{s}}$. The part (a) is called the **Lebesgue decomposition** of λ. The function h is called the **Radon-Nikodym derivative** of λ_{a} with respect to μ, we sometimes write $d\lambda_{\mathrm{a}} = h\,d\mu$.

In particular, if $\lambda \ll \mu$, then $\lambda = h\,d\mu$ for some $h \in L^1_\mu$.

The Lebesgue-Radon-Nikodym theorem can be extended to the case where both μ and λ are positive σ-finite measures. But we cannot go beyond σ-finiteness, as the following exampke shows:

Example: let $X = [0,1]$, μ the Lebesgue measure, and λ the counting measure on X. Then λ cannot be decomposed as $\lambda_{\mathrm{a}} + \lambda_{\mathrm{s}}$. Moreover, we have $\mu \ll \lambda$ but there is no $h \in L^1_\lambda$ such that $d\mu = h\,d\lambda$.

Theorem 9.13. *Let μ be a positive measure and λ a complex measure on $(X, \mathfrak{M})$. Then the following are equivalent:*

(a) $\lambda \ll \mu$*;*

(b) $\forall \varepsilon > 0\ \exists \delta > 0\colon\ \mu(E) < \delta \Longrightarrow |\lambda(E)| < \varepsilon$*.*

The property (b) justifies the name *absolute continuity*.

If λ is a positive but not finite measure, then (a) and (b) are not equivalent. For example, let $X = (0,1)$, μ the Lebesgue measure, and $\lambda(E) = \int_E t^{-1}\,dt$. Then $\lambda \ll \mu$, but the property (b) does not hold.

Theorem 9.14 (Polar representation)**.** *Let μ be a complex measure on $(X, \mathfrak{M})$. Then there is a measurable function $h\colon X \to \mathbb{C}$ such that $|h| = 1$ and $d\mu = h\,d|\mu|$.*

Theorem 9.15. *Let μ be a positive measure on $(X, \mathfrak{M})$. Suppose $g \in L^1_\mu$ and $d\lambda = g\,d\mu$. Then $d|\lambda| = |g|\,d\mu$.*

Theorem 9.16 (Hahn decomposition)**.** *Let μ be a real-valued measure on $(X, \mathfrak{M})$. Then there is a decomposition $X = A \cup B$, $A \cap B = \emptyset$, and*

$$\mu^+(E) = \mu(A \cap E), \qquad \mu^-(E) = \mu(B \cap E)$$

for any $E \in \mathfrak{M}$.

Exercise 60. Let f be a real-valued and integrable function on $[0,1]$. Prove that there exists $c \in [0,1]$ such that

$$\int_{[0,c]} f\,d\mathbf{m} = \int_{[c,1]} f\,d\mathbf{m}.$$

EXERCISE 61. Let $f, f_k \in L^1_m[0,1]$, for $k = 1, 2, \ldots$. Suppose $f_k(x) \to f(x)$ a.e. on $[0,1]$, and $\int_{[0,1]} |f_k|\, dm \to \int_{[0,1]} |f|\, dm$. Show that

$$\int_{[0,1]} |f_k - f|\, dm \to 0.$$

Hint: look at $|f| + |f_k| - |f - f_k|$.

EXERCISE 62. Do the previous exercise with L^1 replaced by L^2.

Let X and Y be vector spaces with norms and $\Lambda\colon X \to Y$ a linear mapping.

DEFINITION 9.17. The **norm** of Λ is

$$\begin{aligned} \|\Lambda\| &= \sup\{\|\Lambda x\| \colon x \in X, \quad \|x\| = 1\} \\ &= \sup\{\|\Lambda x\|/\|x\| \colon x \in X, \quad x \neq 0\} \end{aligned}$$

We say that Λ is **bounded** if $\|\Lambda\| < \infty$.

Lemma 9.18. *We have* $\|\Lambda\| = \inf\{c > 0 \colon \|\Lambda x\| \leq c\|x\| \quad \forall x \in X\}$*; furthermore,* inf *can be replaced with* min*.*

Theorem 9.19. *The following conditions are equivalent:*

(a) Λ *is bounded;*

(b) Λ *is continuous on* X*;*

(c) Λ *is continuous at some point* $x \in X$*.*

DEFINITION 9.20. In a special case, where $Y = \mathbb{C}$, we deal with linear **functionals** $L\colon X \to \mathbb{C}$ and call

$$X^* = \{\text{all bounded linear functionals} \quad L\colon X \to \mathbb{C}\}$$

the **dual space** (to X). It is a vector space with norm $\|L\|$.

Consider $L^p_\mu(X)$ on a measure space $(X, \mathfrak{M}, \mu)$ with $1 \leq p \leq \infty$. Let q be a conjugate exponent. Given a function $g \in L^q_\mu(X)$, we construct a linear functional $\Phi_g\colon L^p_\mu(X) \to \mathbb{C}$ by

$$\Phi_g(f) = \int_X f g\, d\mu$$

It is bounded (by Hölder inequality) and $\|\Phi_g\| \leq \|g\|_q$ (here $\|g\|_q$ denotes the norm in the L^q space).

Theorem 9.21. *Let μ be a finite or σ-finite measure and $1 \leq p < \infty$. Then for every bounded linear functional $\Phi\colon L^p_\mu(X) \to \mathbb{C}$ there exists a unique $g \in L^q_\mu(X)$ (up to equivalence) such that*

$$\Phi(f) = \int_X fg\,d\mu \qquad \forall f \in L^p_\mu(X),$$

i.e. $\Phi = \Phi_g$; furthermore, $\|\Phi\| = \|g\|_q$.

In other words, L^q_μ is isometrically isomorphic to the dual space $L^p_\mu(X)^*$.
For $1 < p < \infty$, the theorem holds without σ-finiteness.
This theorem does not hold for $p = \infty$, see the next exercise.

Exercise 63. Let $X = [0,1]$ and μ the Lebesgue measure. Show that $L^\infty(X)^* \supset L^1(X)$, but $L^\infty(X)^* \neq L^1(X)$ (in the sense $g \to \Phi_g$). (Hint: Use the following consequence of the Hahn-Banach theorem: If X is a Banach space and $A \subset X$ is a closed subspace of X, with $A \neq X$, then there exists $f \in X^*$ with $f \neq 0$, and $f(x) = 0$ for all $x \in A$.)

Exercise 64. Suppose μ is a positive measure on X and $\mu(X) < \infty$. Let $f \in L^\infty_\mu$ and $\|f\|_\infty > 0$. Define

$$\alpha_n = \int_X |f|^n\,d\mu \qquad (n = 1, 2, 3, \dots).$$

Prove that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

(Hint: Start with $\|f\|_\infty = 1$, use Hölder inequality and the result of Exercise 51 (e).)

10 Differentiation

Definition 10.1. Let μ be a complex measure on $\mathbb{R}$ (with Borel σ-algebra). Then $F(x) = \mu\big((-\infty, x)\big)$ is called the **distribution function** of the measure μ.

We denote by $\mathbf{m}$ the Lebesgue measure on $\mathbb{R}$ (and on $\mathbb{R}^k$).

Theorem 10.2. *The following two conditions are equivalent:*

(a) *F is differentiable at $x \in \mathbb{R}$ and $F'(x) = A$;*

(b) $\forall \varepsilon > 0\ \exists \delta > 0$: $\left|\frac{\mu(I)}{\mathbf{m}(I)} - A\right| < \varepsilon$ *for every open interval $I \subset \mathbb{R}$ such that $x \in I$ and $\mathbf{m}(I) < \delta$.*

Motivated by this we will define derivatives of measures in $\mathbb{R}^k$.

Definition 10.3. For $x \in \mathbb{R}^k$ and $r > 0$ we denote by

$$B(x, r) = \{y \in \mathbb{R}^k \colon |y - x| < r\}$$

the **open ball** of radius r centered on x. If μ is a complex measure on $\mathbb{R}^k$ (with Borel σ-algebra), then we denote

$$(Q_r\mu)(x) = \frac{\mu\big(B(x,r)\big)}{\mathbf{m}\big(B(x,r)\big)}$$

and define the **symmetric derivative** of μ at x by

$$(D\mu)(x) = \lim_{r\to 0}(Q_r\mu)(x)$$

(if the limit exists).

Exercise 65. Let μ be a complex Borel measure on $\mathbb{R}$ and assume that its symmetric derivative $(D\mu)(x)$ exists at some $x_0 \in \mathbb{R}$. Does it follow that its distribution function $F(x) = \mu((-\infty, x))$ is differentiable at x_0?

To study derivatives of measures, we introduce more definitions.

Definition 10.4. If μ is a complex measure on $\mathbb{R}^k$ (with Borel σ-algebra), then the **maximal function** $M\mu \colon \mathbb{R}^k \to [0, \infty]$ defined by

$$(M\mu)(x) = \sup_{0<r<\infty} (Q_r|\mu|)(x).$$

Lemma 10.5. *$M\mu$ is lower semicontinuous, i.e. the set $\{x \colon M\mu(x) > a\}$ is open for every $a \in \mathbb{R}$.*

Lemma 10.6 (Covering)**.** *Let $W = \cup_{i=1}^N B(x_i, r_i)$ be a finite union of open balls. Then there is $S \subset \{1, 2, \ldots, N\}$ such that*

(a) *the balls (x_i, r_i), $i \in S$ are disjoint;*

(b) *$W \subset \cup_{i\in S} B(x_i, 3r_i)$;*

(c) *$\mathbf{m}(W) \le 3^k \sum_{i\in S} \mathbf{m}\big(B(x_i, r_i)\big)$.*

Theorem 10.7. *Let μ be a complex measure on $\mathbb{R}^k$. Then $\forall \lambda > 0$*

$$\mathbf{m}(M\mu > \lambda) \leq \frac{3^k \|\mu\|}{\lambda}$$

where $\|\mu\| = |\mu|(\mathbb{R}^k)$.

In other words, the maximal function cannot be large on a large set.

DEFINITION 10.8. **Weak** L^1 is the space of measurable functions

$$L^1_W(\mathbb{R}^k) = \{f\colon \mathbb{R}^k \to \mathbb{C} \text{ such that } \lambda\, \mathbf{m}\{|f| > \lambda\} \text{ is bounded on } \lambda \in (0,\infty)\}$$

Lemma 10.9. *We have $L^1_{\mathbf{m}}(\mathbb{R}^k) \subset L^1_W(\mathbb{R}^k)$.*

Note: there are functions in $L^1_W(\mathbb{R}^k)$ that are not in $L^1_{\mathbf{m}}(\mathbb{R}^k)$. For example, $f = 1/x$ for $k = 1$.

DEFINITION 10.10. For every $f \in L^1_W(\mathbb{R}^k)$ define the **maximal function** $Mf\colon \mathbb{R}^k \to [0,\infty]$ by

$$(Mf)(x) = \sup_{0<r<\infty} \frac{1}{\mathbf{m}(B(x,r))} \int_{B(x,r)} |f|\, d\mathbf{m}.$$

Note that $\mathbf{m}(B(x,r)) = \mathbf{m}(B(0,r))$ does not depend on x.

We have $Mf = M\mu$, where $d\mu = f\, d\mathbf{m}$. Thus theorem 10.7 states that the 'maximal function' M induces an operator $L^1_{\mathbf{m}}(\mathbb{R}^k) \to L^1_W(\mathbb{R}^k)$, which is bounded (its bound is $\leq 3^k$): for every $f \in L^1_{\mathbf{m}}(\mathbb{R}^k)$ and every $\lambda > 0$

$$\mathbf{m}\{Mf > \lambda\} \leq 3^k \lambda^{-1} \|f\|_1.$$

DEFINITION 10.11. If $f \in L^1_{\mathbf{m}}(\mathbb{R}^k)$, then any point $x \in \mathbb{R}^k$ for which

$$\lim_{r\to 0} \frac{1}{\mathbf{m}(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, d\mathbf{m}(y) = 0$$

is called a **Lebesgue point** of f.

Roughly speaking, x is a Lebesgue point if f does not oscillate too much near x, in average sense.

Lemma 10.12. *If f is continuous at x, then x is a Lebesgue point of f.*

Lemma 10.13. *If x is a Lebesgue point of f, then*

$$f(x) = \lim_{r\to 0} \frac{1}{\mathbf{m}(B(x,r))} \int_{B(x,r)} f\, d\mathbf{m}$$

(but the converse is not true).

Exercise 66. Let $f \in L^1_{\mathbf{m}}(\mathbb{R}^k)$. Show that $|f(x)| \le (Mf)(x)$ at every Lebesgue point x of f.

Exercise 67. Let

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \ \text{ or } \ x > 1 \\ 1 & \text{if} \quad 0 < x < 1 \end{cases}.$$

Is it possible to define $f(0)$ and $f(1)$ such that 0 and 1 become Lebesgue points of f?

Exercise 68 (Bonus). Construct a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and 0 is a Lebesgue point of f, but for every $\varepsilon > 0$

$$\mathbf{m}\big\{x \in \mathbb{R}\colon |x| < \varepsilon \ \text{ and } \ |f(x)| \ge 1\big\} > 0,$$

i.e. f is essentially discontinuous at 0.

Theorem 10.14. *If $f \in L^1(\mathbb{R}^k)$, then almost every point $x \in \mathbb{R}^k$ is a Lebesgue point of f.*

Theorem 10.15. *Let μ be a complex Borel measure on $\mathbb{R}^k$ such that $\mu \ll \mathbf{m}$. Then the Radon-Nokodym derivative $f = d\lambda/d\mathbf{m}$ is equal to the symmetric derivative of μ, i.e. $f = D\mu$ $\mathbf{m}$-a.e.*

Next we explore some other ways of computing symmetric derivatives.

Definition 10.16. Let $x \in \mathbb{R}^k$. We say that Borel sets $E_i \subset \mathbb{R}^k$ **shrink nicely to** x if there exists $\alpha > 0$ such that for some sequence of balls $B(x, r_i)$ with $r_i \to 0$ we have $E_i \subset B(x, r_i)$ and $m(E_i) \ge \alpha \mathbf{m}(B(x, r_i))$ for all i.

Examples: any sequence of intervals $I_i \supset x$ in $\mathbb{R}^k$ such that $\mathbf{m}(I_i) \to 0$ shrinks to x nicely. More generally: any sequence of balls or cubes in $\mathbb{R}^k$ must shrink nicely. But the sequence of rectangles $[-\frac{1}{i}, \frac{1}{i}] \times [-\frac{1}{i^2}, \frac{1}{i^2}]$ does not shrink nicely to 0.

Theorem 10.17. *If $f \in L^1(\mathbb{R}^k)$ and $x \in \mathbb{R}^k$ is a Lebesgue point of f, then for any sequence $\{E_i\}$ that shrinks nicely to x we have*

$$f(x) = \lim_{i\to\infty} \frac{1}{\mathbf{m}(E_i)} \int_{E_i} f\, d\mathbf{m}.$$

Recall the Fundamental Theorem of Calculus:

(a) if $f\colon [a,b] \to \mathbb{R}$ is continuous and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ for all $x \in (a,b)$.

(b) if $f\colon [a,b] \to \mathbb{R}$ is continuously differentiable, then

$$f(x) - f(a) = \int_a^x f'(t)\,dt$$

for all $x \in [a,b]$.

Note that in both parts the Riemannian integral is used. Our goal is to extend this theorem to Lebesgue integrable functions.

Theorem 10.18 (FTC in L^1, easy part)**.** *If $f \in L^1(\mathbb{R})$ and $F(x) = \int_{(-\infty,x]} f\,d\mathbf{m}$ for $x \in \mathbb{R}$, then $F'(x) = f(x)$ at every Lebesgue point x of f.*

Definition 10.19. Let $E \subset \mathbb{R}^k$ be a measurable set and $x \in \mathbb{R}^k$. Then

$$\lim_{r\to 0} \frac{\mathbf{m}(E \cap B(x,r))}{\mathbf{m}(B(x,r))}$$

is the **metric density** of E at x (if the limit exists).

Corollary 10.20. *For every measurable set $E \subset \mathbb{R}^k$ the metric density is 1 at a.e. point $x \in E$ and 0 at a.e. point $x \in E^c$.*

Corollary 10.21. *If $\varepsilon > 0$, then there is no measurable set $E \subset \mathbb{R}$ such that*

$$\epsilon < \frac{\mathbf{m}(E \cap I)}{\mathbf{m}(I)} < 1 - \varepsilon$$

for every finite nontrivial interval $I \subset \mathbb{R}$.

In fact, if $\epsilon < \frac{\mathbf{m}(E\cap I)}{\mathbf{m}(I)}$ for every finite nontrivial interval $I \subset \mathbb{R}$, then $\mathbf{m}(E^c) = 0$. And if $\frac{\mathbf{m}(E\cap I)}{\mathbf{m}(I)} < 1 - \varepsilon$ for every finite nontrivial interval $I \subset \mathbb{R}$, then $\mathbf{m}(E) = 0$.

Our next goal is to extend to L^1 the second part of the FTC. We want to find a class of functions $f\colon [a,b] \to \mathbb{R}$ such that

$$f(x) - f(a) = \int_{[a,x]} f'\,d\mathbf{m} \qquad \forall x \in [a,b] \tag{FTC-2}$$

Obviously, f must be differentiable (at least a.e.) and f' must be integrable (i.e. $f' \in L^1([a,b])$). But is this enough?

Answer: No! Counterexample: Cantor function (devil's staircase) constructed via the middle-third Cantor set. This example shows that (FTC-2) may fail if f experiences 'rapid growth on tiny sets'.

Definition 10.22. A function $f\colon [a,b] \to \mathbb{C}$ is said to be **absolutely continuous (a.c.)** on an interval $[a,b]$ if $\forall \varepsilon > 0\ \exists \delta > 0$ such that if $n \in \mathbb{N}$ and

$$(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$$

are disjoint subintervals of $[a,b]$, then

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta \quad \Longrightarrow \quad \sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \varepsilon.$$

Note: if f is absolutely continuous, then f is continuous, even uniformly continuous (but not vice versa).

Lemma 10.23. *If $f\colon [a,b] \to \mathbb{C}$ is differentiable at almost all $x \in [a,b]$, $f' \in L^1([a,b])$, and (FTC-2) holds, then f is absolutely continuous.*

We will show that the converse is also true.

Theorem 10.24. *Let $I = [a,b]$ and $f\colon I \to \mathbb{R}$ be a continuous and nondecreasing function. Then the following properties are equivalent:*

(a) *f is absolutely continuous;*

(b) *f maps sets of measure zero to sets of measure zero;*

(c) *f is differentiable a.e. on I, $f' \in L^1(I)$ and (FTC-2) holds.*

Definition 10.25. The **total variation function** V_a^x of a function $f\colon [a,b] \to \mathbb{C}$ is defined by

$$V_a^x = \sup \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$$

where the supremum is taken over all n and all ordered sequences

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = x.$$

If $V_a^b < \infty$, then f is said to be of **bounded variation** on $[a,b]$, denoted by $f \in BV[a,b]$, and the value of V_a^b is called the **total variation** of f.

Lemma 10.26. *Suppose $f(x)$ is of bounded variation on $[a,b]$. Then*

(a) *V_a^x is non-decreasing in x;*

(b) *if f is real-valued, then $V+f$ and $V-f$ are non-decreasing;*

Corollary 10.27. *Let $f\colon [a,b] \to \mathbb{R}$ be of bounded variation on $[a,b]$. Then there are (strictly) monotonically increasing functions $u, v\colon [a,b] \to \mathbb{R}$ such that $f = u - v$.*

Lemma 10.28. *Suppose $f(x)$ is absolutely continuous on $[a,b]$. Then*

(a) *f is of bounded variation on $[a,b]$;*

(b) *V, $V+f$, and $V-f$ are absolutely continuous on $[a,b]$.*

Theorem 10.29. *A function $f\colon [a,b] \to \mathbb{C}$ is absolutely continuous if and only if f is differentiable for a.e. $x \in [a,b]$, $f' \in L^1([a,b])$, and (FTC-2) holds.*

This theorem answers the question posed around (FTC-2).

Example: Consider a function

$$f(x) = \begin{cases} x^2 \cos \frac{\pi}{x^2} & \text{for } 0 < x \le 1 \\ 0 & \text{for } x = 0 \end{cases}$$

It is differentiable at every point $x \in [0,1]$ (including $x = 0$), but f' does not belong to $L^1([0,1])$. So this function is not absolutely continuous.

The following theorem can be stated without proof:

Theorem 10.30. *Let $f\colon [a,b] \to \mathbb{C}$ be differentiable at every point $x \in [a,b]$ and $f' \in L^1([a,b])$. Then (FTC-2) holds.*

Note: the differentiability at *every* point $x \in [a,b]$ is assumed.

Example: Consider a function

$$f(x) = \begin{cases} x^2 \cos \frac{\pi}{x} & \text{for } 0 < x \le 1 \\ 0 & \text{for } x = 0 \end{cases}$$

It is differentiable at every point $x \in [0,1]$ (including $x = 0$) and f' is bounded, hence $f' \in L^1([0,1])$. So this function, unlike the one in the previous example, is absolutely continuous.

EXERCISE 69. Let $\alpha > 0$, $\beta > 0$ and

$$f(x) = \begin{cases} x^\alpha \cos \frac{\pi}{x^\beta} & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0 \end{cases}$$

Show that if $\alpha > 1$ and $\alpha > \beta$, then f is absolutely continuous on $[0, 1]$. For an extra credit, show that the assumption $\alpha > 1$ is redundant.

EXERCISE 70. Let f, g be absolutely continuous on $[a, b]$. Show that
(a) fg is absolutely continuous on $[a, b]$.
(b) if $f(x) \neq 0$ for all $x \in [a, b]$, then $1/f$ is absolutely continuous on $[a, b]$.
(c) Does (a) remain true if $[a, b]$ is replaced by $\mathbb{R}$?

EXERCISE 71. A function $f \colon [a, b] \to \mathbb{C}$ is said to be Lipschitz continuous if $\exists L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in [a, b]$. Prove that if f is Lipschitz continuous, then f is absolutely continuous and $|f'| \leq L$ a.e. Conversely, if f is absolutely continuous and $|f'| \leq L$ a.e., then f is Lipschitz continuous (with that constant L).

Proposition 10.31. *Let $f \colon [a, b] \to \mathbb{R}$ be absolutely continuous. Then it maps sets of measure zero to sets of measure zero.*

Note: the converse is not true, even if f is continuous or even differentiable (see the first example above). Also, this proposition does not extend to functions of bounded variations (a counterexample: devil's staircase maps the Cantor set onto the entire unit interval).

Corollary 10.32. *Let $f \colon [a, b] \to \mathbb{R}$ be absolutely continuous. Then it maps Lebesgue measurable sets to Lebesgue measurable sets.*

The following theorem can be stated without proof:

Theorem 10.33. *Let $f \colon [a, b] \to \mathbb{R}$ be monotonically increasing. Then it is differentiable a.e. and $f(b) - f(a) \geq \int_{[a,b]} f' \, d\mathbf{m}$.*

Note: a strict inequality is possible (example: devil's staircase).

EXERCISE 72. Let $f \colon [a, b] \to \mathbb{R}$ be absolutely continuous. Prove that $V_a^x \leq \int_{[a,x]} |f'| \, d\mathbf{m}$. For an extra credit: is the equality always true?

EXERCISE 73. Let $f \colon [0, 1] \to \mathbb{R}$ be absolutely continuous on $[\delta, 1]$ for each $\delta > 0$ and continuous and of bounded variation on $[0, 1]$. Prove that f is absolutely continuous on $[0, 1]$. [Hint: use the result of the previous exercise to verify that $f' \in L^1([0, 1])$.]

Exercise 74 (Bonus). Let $f\colon [a,b] \to \mathbb{C}$ be absolutely continuous. Prove that $|f|$ is absolutely continuous and

$$\Big|\frac{d}{dx}|f(x)|\Big| \leq |f'(x)|$$

for a.e. $x \in [a,b]$. [Hint: first, use triangle inequality in the form $\big||p| - |q|\big| \leq |p - q|$, for complex numbers p, q, to show that $|f|$ is AC. Then prove the above inequality for $x \in [a,b]$ that are Lebesgue points for both f' and $|f|'$; use Theorem 10.17.]

Let $A\colon \mathbb{R}^k \to \mathbb{R}^k$ be a linear transformation defined by a $k \times k$ matrix, which we also call A. Then for any measurable set $E \subset \mathbb{R}^k$ we have

$$\mathbf{m}(A(E)) = c\,\mathbf{m}(E)$$

where the scaling factor is $c = |\det A|$. In particular, every isometry (translation, rotation, reflection) preserves the Lebesgue measure.

For a (nonlinear) transformation $T\colon V \to \mathbb{R}^k$ defined on an open set $V \subset \mathbb{R}^k$ we define its derivative $T'(x) = A$ at a point $x \in V$ by

$$\lim_{h\to 0} \frac{\|T(x+h) - T(x) - Ah\|}{\|h\|} = 0$$

(where of course $h \in \mathbb{R}^k$), if the limit exists. Note that the derivative $T'(x)$ is a $k \times k$ matrix. (If you define the transformation T coordinate-wise, by k functions of k variables, then $T'(x)$ will be the matrix of their partial derivatives.)

Lemma 10.34. *Let $T\colon V \to \mathbb{R}^k$ be differentiable at $x \in V$. Then*

$$\lim_{r\to 0} \frac{\mathbf{m}(T(B(x,r)))}{\mathbf{m}(B(x,r))} = |\det T'(x)|$$

The following is a (global) change of variables rule:

Theorem 10.35. *Let $T\colon V \to \mathbb{R}^k$ be differentiable at every point $x \in V$ and one-to-one. Then for any measurable function $f\colon \mathbb{R}^k \to [0, +\infty]$*

$$\int_{T(V)} f\, d\mathbf{m} = \int_V (f \circ T)\, |\det T'|\, d\mathbf{m}$$

The factor $\det T'$ is called the Jacobian (of the transformation T).

Note: a continuous (not differentiable) transformation may map measurable sets into nonmeasurable sets. Example: devil's staircase.

11 Integration on product spaces

Definition 11.1. Let X and Y be two sets. Its **Cartesian product** $X \times Y$ is the set of all (ordered) pairs (x, y), with $x \in X$ and $y \in Y$. If $A \subset X$ and $B \subset Y$, then the set $A \times B \subset X \times Y$ is called a **rectangle**.

Definition 11.2. Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be two measurable spaces. For every $A \in \mathfrak{M}$ and $B \in \mathfrak{N}$ the set $A \times B$ is called a **measurable rectangle** in $X \times Y$. Any finite union $E = R_1 \cup R_2 \cup \cdots \cup R_n$ of disjoint measurable rectangles is called an **elementary set**. The collection of elementary sets is denoted by $\mathcal{E}$.

Lemma 11.3. *$\mathcal{E}$ is an algebra, i.e. finite unions, intersections and differences of elementary sets are elementary sets.*

We denote by $\mathfrak{M} \times \mathfrak{N}$ the (minimal) σ-algebra in $X \times Y$ generated by elementary sets (equivalently, by measurable rectangles).

Definition 11.4. Let $E \subset X \times Y$ and $x \in X$, $y \in Y$. Then

$$E_x = \{y' \colon (x, y') \in E\} \subset Y$$

is called the x-**section** of E and

$$E^y = \{x' \colon (x', y) \in E\} \subset X$$

is called the y-**section** of E.

Theorem 11.5. *If $E \in \mathfrak{M} \times \mathfrak{N}$, then $E_x \in \mathfrak{N}$ and $E^y \in \mathfrak{M}$ for every $x \in X$ and $y \in Y$.*

Definition 11.6. A **monotone class** $\mathcal{M}$ is a collection of sets with the following properties:

(i) $A_i \in \mathcal{M},\ A_i \subset A_{i+1}\ (i = 1, 2, \ldots) \implies \cup_{i=1}^{\infty} A_i \in \mathcal{M}$;

(ii) $B_i \in \mathcal{M},\ B_i \supset B_{i+1}\ (i = 1, 2, \ldots) \implies \cup_{i=1}^{\infty} B_i \in \mathcal{M}$.

Note: σ-algebras are monotone classes, but not vice versa. The intersection of monotone classes is a monotone class.

Lemma 11.7. *If X is a set and $\mathcal{F}$ a collection of subsets of X, then there exists a smallest monotone class $\mathcal{M}$ that contains $\mathcal{F}$.*

Theorem 11.8. *The σ-algebra $\mathfrak{M} \times \mathfrak{N}$ is the smallest monotone class containing $\mathcal{E}$.*

For any function $f\colon X \times Y \to Z$ and $x \in X$ we denote by f_x a function on Y defined by $f_x(y) = f(x, y)$. Similarly, for any $y \in Y$ we denote by f^y a function on X defined by $f^y(x) = f(x, y)$.

Theorem 11.9. *Let Z be a topological space and $f\colon X \times Y \to Z$ a measurable function (with respect to the σ-algebra $\mathfrak{M} \times \mathfrak{N}$). Then*

(i) *f_x is $\mathfrak{N}$-measurable on Y for each $x \in X$;*

(ii) *f^y is $\mathfrak{M}$-measurable on X for each $y \in Y$;*

Next we construct the product of measures.

Theorem 11.10. *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \lambda)$ be two measure spaces and let $Q \in \mathfrak{M} \times \mathfrak{N}$. Define*

$$\begin{aligned} \varphi(x) &= \lambda(Q_x) \quad \textit{for every } x \in X \\ \psi(x) &= \mu(Q^y) \quad \textit{for every } y \in Y \end{aligned}$$

Then φ is $\mathfrak{M}$-measurable, ψ is $\mathfrak{N}$-measurable, and

$$\int_X \varphi \, d\mu = \int_Y \psi \, d\lambda.$$

Definition 11.11. Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \lambda)$ be two measure spaces with σ-finite measures. Then we can define a measure, $\mu \times \lambda$, on $\mathfrak{M} \times \mathfrak{N}$ by

$$(\mu \times \lambda)(Q) = \int_X \lambda(Q_x) \, d\mu(x) = \int_Y \mu(Q^y) \, d\lambda(y).$$

The measure $\mu \times \lambda$ is called the **product** of the measures μ and λ.

Observe that $\mu \times \lambda$ is also a σ-finite measure.
We can rewrite the above formula as

$$(\mu \times \lambda)(Q) = \int_X \left(\int_Y \chi_Q(x, y) \, d\lambda(y) \right) d\mu(x) = \int_Y \left(\int_X \chi_Q(x, y) \, d\mu(x) \right) d\lambda(y).$$

EXERCISE 75. Given measure spaces $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \lambda)$ with σ-finite measures, show that $\mu \times \lambda$ is the unique measure on $\mathfrak{M} \times \mathfrak{N}$ such that

$$(\mu \times \lambda)(A \times B) = \mu(A)\lambda(B)$$

for all measurable rectangles $A \times B$ in $X \times Y$.

Theorem 11.12 (Fubini)**.** *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \lambda)$ be two measure spaces with σ-finite measures and let f be an $(\mathfrak{M}\times\mathfrak{N})$-measurable function on $X \times Y$. Then*

(a) *If $0 \le f \le \infty$ and if*

$$\varphi(x) = \int_Y f_x \, d\lambda, \qquad \psi(y) = \int_X f^y \, d\mu,$$

then φ is $\mathfrak{M}$-measurable, ψ is $\mathfrak{N}$-measurable, and

$$\int_X \varphi \, d\mu = \int_{X\times Y} f \, d(\mu \times \lambda) = \int_Y \psi \, d\lambda. \tag{Fubini}$$

(b) *If f is complex-valued and $\int_X \varphi^* \, d\mu < \infty$, where $\varphi^*(x) = \int_Y |f|_x \, d\lambda$, then $f \in L^1_{\mu\times\lambda}$.*

(c) *If $f \in L^1_{\mu\times\lambda}$, then*

$$\begin{aligned} f_x &\in L^1_\lambda \quad \text{for a.e. } \ x \in X, \\ f^y &\in L^1_\mu \quad \text{for a.e. } \ y \in Y, \\ \varphi(x) &= \int_Y f_x \, d\lambda \in L^1_\mu, \\ \psi(y) &= \int_X f^y \, d\mu \in L^1_\lambda, \end{aligned}$$

and the above equation (Fubini) *holds.*

Note that equation (Fubini) can be written as

$$\begin{aligned} \int_{X\times Y} f \, d(\mu \times \lambda) &= \int_X \left(\int_Y f(x, y) \, d\lambda(y) \right) d\mu(x) \\ &= \int_Y \left(\int_X f(x, y) \, d\mu(x) \right) d\lambda(y) \end{aligned} \tag{DoubleIter}$$

(a double integral is represented by iterated integrals).

Corollary 11.13. *If one iterated integral in* (DoubleIter) *exists, then all the three exist and are equal.*

EXERCISE 76. Let $a_n \geq 0$ for $n = 1, 2, \ldots$, and for $t \geq 0$ let

$$N(t) = \#\{n : a_n > t\}$$

Prove that

$$\sum_{n=1}^{\infty} a_n = \int_0^{\infty} N(t)\, dt$$

For an extra credit, find a formula for $\sum_{n=1}^{\infty} \phi(a_n)$ in terms of $N(t)$ as defined above, if $\phi : [0, \infty) \to [0, \infty)$ is locally absolutely continuous (i.e., AC on every finite interval), non-decreasing, and $\phi(0) = 0$ (Hint: use the above fact with $\phi(x) = x$).

EXERCISE 77. Let $f \in L^1([0, 1])$ and $f \geq 0$. Show that

$$\int_0^1 \frac{f(y)}{|x - y|^{1/2}}\, dy$$

is finite for a.e. $x \in [0, 1]$ and integrable.

EXERCISE 78. Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^{\infty} e^{-xt}\, dt \quad \text{for} \quad x > 0$$

to prove that

$$\lim_{A \to \infty} \int_0^A \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

It is instructive to see a few counterexamples that show that the assumptions of Fubini's theorem cannot be dispensed with.

First, let $X = Y = \mathbb{N}$ and $\mu = \lambda$ counting measure. Then $\mu \times \lambda$ is the counting measure on $X \times Y$. Define a function $f(x, y)$ on $X \times Y$ as follows: $f(i, j) = a_{ij}$ for $i, j \geq 1$, where

$$a_{ij} = \begin{cases} +1 & \text{if} \ \ i = j \\ -1 & \text{if} \ \ i = j + 1 \\ \ \ 0 & \text{otherwise} \end{cases}$$

Then

$$\int_X \left(\int_Y f(x, y)\, d\lambda(y) \right) d\mu(x) = \sum_i \sum_j a_{ij} = 1$$

but

$$\int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\lambda(y) = \sum_j \sum_i a_{ij} = 0.$$

Incidentally, we found a sequence $\{a_{ij}\}$ such that $\sum_i \sum_j a_{ij} \neq \sum_j \sum_i a_{ij}$. The reason why Fubini's theorem does not apply is that $f \notin L^1_{\mu\times\lambda}$.

See a Lebesgue-measure version of this example in Rudin (page 166).

The product measure $\mu \times \lambda$ is not necessarily complete. In fact, if there is a non-empty null set $A \subset X$ and a non-measurable set $B \subset Y$, then $A \times B$ is not measurable but it is a subset of a null set $A \times Y$. In particular, if $X = Y = \mathbb{R}$ and $\mu = \lambda = \mathbf{m}$ the Lebesgue measure, then $\mathbf{m} \times \mathbf{m}$ is not a complete measure, thus it is not a Lebesgue measure on $\mathbb{R}^2$.

Theorem 11.14. *Let $\mathbf{m}_k$ denote the Lebesgue measure on $\mathbb{R}^k$. Then the completion of $\mathbf{m}_s \times \mathbf{m}_t$ is the Lebesgue measure $\mathbf{m}_{s+t}$.*

There is an alternative version of Fubini's theorem, adapted to complete measures (in particular, to Lebesgue measures on $\mathbb{R}^k$):

Theorem 11.15. *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \lambda)$ be two measure spaces with complete σ-finite measures. Let $\big(X \times Y, (\mathfrak{M} \times \mathfrak{N})^*, (\mu \times \lambda)^*\big)$ be the completion of the measure $\mu \times \lambda$. Let f be an $(\mathfrak{M} \times \mathfrak{N})^*$-measurable function on $X \times Y$. Then f_x is $\mathfrak{N}$-measurable for almost every $x \in X$, f^y is $\mathfrak{M}$-measurable for almost every $y \in Y$, and all the other conclusions of Fubini's theorem hold.*

The following interesting lemma is used in the proof:

Lemma 11.16. *Let ν be a positive measure on a σ-algebra $\mathfrak{M}$ in X, and $(X, \mathfrak{M}^*, \mu^*)$ be its completion. Then for every $\mathfrak{M}^*$-measurable function f there exists a $\mathfrak{M}$-measurable function g such that $f = g$ ν-almost everywhere.*

Definition 11.17. Let $f\colon \mathbb{R} \to \mathbb{C}$ and $g\colon \mathbb{R} \to \mathbb{C}$ be two Lebesgue measurable functions. Then their **convolution** $h = f * g$ is defined by

$$h(x) = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

(whenever the integral exists).

Theorem 11.18. *Let $f, g \in L^1(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty} |f(x-t)\,g(t)|\,dt < \infty$$

for almost every $x \in \mathbb{R}$ and $h = f * g \in L^1(\mathbb{R})$ with

$$\|h\|_1 \le \|f\|_1 \|g\|_1 \qquad \text{(Young's inequality)}$$

We note that it is not always true that $fg \in L^1(\mathbb{R})$. For example, consider $f = g = x^{-1/2}\chi_{[0,1]}$, then $f, g \in L^1(\mathbb{R})$ but $fg \notin L^1(\mathbb{R})$.

If we assume that $f, g \ge 0$, then $\|h\|_1 = \|f\|_1 \|g\|_1$.

Exercise 79. Suppose $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ for some $1 \le p \le \infty$. Show that $f * g$ exists at a.e. $x \in \mathbb{R}$ and $f * g \in L^p(\mathbb{R})$, and prove that

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

Hints: the case $p = \infty$ is simple and can be treated separately. If $p < \infty$, then use Hölder inequality and argue as in the proof of the previous theorem.

Exercise 80. Let $f \in L^1(\mathbb{R})$ and

$$g(x) = \int_{\mathbb{R}} f(y)\, e^{-(x-y)^2}\, dy.$$

Show that $g \in L^p(\mathbb{R})$, for all $1 \le p \le \infty$, and estimate $\|g\|_p$ in terms of $\|f\|_1$. You can use the following standard fact: $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

Exercise 81 (Bonus). Let $E = [1, \infty)$ and $f \in L^2_{\mathbf{m}}(E)$. Also assume that $f \ge 0$ a.e. and define

$$g(x) = \int_E f(y)\, e^{-xy}\, dy.$$

Show that $g \in L^1(E)$ and

$$\|g\|_1 \le c\, \|f\|_2$$

for some $c < 1$. Estimate the minimal value of c the best you can.

Definition 11.19. Let $(X, \mathfrak{M}, \mu)$ be a measure space with a σ-finite measure and $f\colon X \to [0, \infty]$ a measurable function. Then

$$g(t) = \mu(\{x \in X\colon f(x) > t\}) = \mu\{f > t\}$$

is called the **distribution function** of f.

(In probability theory, where $\mu(X) = 1$, the distribution function is defined by $F(t) = \mu\{f \le t\} = 1 - g(t)$.)

Note that the distribution function is monotonic non-increasing, thus it is a Borel measurable function.

Theorem 11.20. *Let f and μ be as above. Suppose $\varphi\colon [0,\infty] \to [0,\infty]$ is a monotonic non-decreasing function that is absolutely continuous on every finite interval $[0,T]$ and satisfies $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \varphi(\infty)$. Then*

$$\int_X (\varphi \circ f)\, d\mu = \int_0^\infty \mu\{f > t\}\varphi'(t)\, dt.$$

In particular, if $\varphi(t) = t$ we obtain a useful formula

$$\int_X f\, d\mu = \int_0^\infty \mu\{f > t\}\, dt,$$

which is sometimes given as a definition of the Lebesgue integral.

Exercise 82. Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Prove that
(a) $A = \{(x,y) \in \mathbb{R}^2 \mid y < f(x)\}$ is Lebesgue measurable (in the two-dimensional sense)
(b) Let $f \ge 0$. Is it always true that $\int_{\mathbb{R}} f\, d\mathbf{m}$ equals the Lebesgue measure of A?
(c) $\{(x,y) \in \mathbb{R}^2 \mid y = f(x)\}$ is a null set.

Definition 11.21. For every Lebesgue measurable function $f\colon \mathbb{R}^k \to \mathbb{C}$ define the **maximal function** $Mf\colon \mathbb{R}^k \to [0,\infty]$ by

$$(Mf)(x) = \sup_{0<r<\infty} \frac{1}{\mathbf{m}(B(x,r))} \int_{B(x,r)} |f|\, d\mathbf{m}$$

whenever the integral exists.

Note: earlier we defined this function only for $f \in L^1(\mathbb{R}^k)$, and in that case we proved $Mf \in L^1_W(\mathbb{R}^k)$.

Theorem 11.22. *If $f \in L^1(\mathbb{R}^k)$ and $Mf \in L^1(\mathbb{R}^k)$, then $f = 0$ a.e.*

Exercise 83. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be such that f_x is Borel-measurable for every $x \in \mathbb{R}$ and f^y is continuous for every $y \in \mathbb{R}$. Prove that f is Borel-measurable. (See hint on p. 176 in Rudin.)

Theorem 11.23 (Hardy–Littlewood)**.** *Let $1 < p \le \infty$. If $f \in L^p(\mathbb{R}^k)$, then $Mf \in L^p(\mathbb{R}^k)$.*