MA 646-2F (Real Analysis), Dr. Chernov | Midterm test
The test consists of 10 problems. | Thu, Mar 14, 2013

Part A. Does the following exist? (If not, refer to a theorem or briefly sketch a proof; if yes, briefly describe an example.)

1. A positive Borel measure μ on $\mathbb{R}$ which is *not* a complex measure.
2. A complex Borel measure μ on $\mathbb{R}$ which is *not* of the form $\mu(A) = \int_A f\, d\mathbf{m}$ for some $f \in L^1(\mathbb{R})$ and all Borel sets A.
3. Two non-zero mutually singular complex Borel measures, $\mu \perp \lambda$, on $\mathbb{R}$ that are both absolutely continuous with respect to the Lebesgue measure, i.e. such that $\mu \ll \mathbf{m}$ and $\lambda \ll \mathbf{m}$.
4. Two non-zero complex Borel measures, μ and λ, on $\mathbb{R}$ such that
$$\mu \ll \mathbf{m}, \qquad \lambda \perp \mathbf{m}, \qquad \text{and} \qquad \mu \ll \lambda$$
5. An open set $V \subset [0,1]$ such that $\mathbf{m}(\bar{V}) > \mathbf{m}(V)$. ($\bar{V}$ denotes the closure of V.)

Part B. Provide solutions to the following problems. Justify your answer.

6. Let $f, g \in L^1_{\mathbf{m}}([a,b])$ and $\int_{[a,x]} f\, d\mathbf{m} = \int_{[a,x]} g\, d\mathbf{m}$ for every $x \in [a,b]$. Is it true that $f(x) = g(x)$ for almost every $x \in [a,b]$? (Hint: consider the function $h = f - g$.)

7. Is the function $f(x) = \sqrt{x}$ absolutely continuous on the interval $[0,1]$? (Hint: use the FTC in L^1.)

8. Does there exist an absolutely continuous function f on $[0,1]$ and a sequence of Borel sets $E_n \subset [0,1]$ such that $\mathbf{m}(E_n) > 0$ and
$$\frac{\mathbf{m}(f(E_n))}{\mathbf{m}(E_n)} > n$$
for every $n \geq 1$?

9. Identify all Lebesgue points of the Dirichlet function $f = \chi_{\mathbb{Q}\cap[0,1]}$.

10. Given a function $f(x) = 2x^3 - 3x^2$, find its total variation on the interval $[-1,2]$, i.e., find V_{-1}^2. Justify your answer.