MA 646-2F (Real Analysis), Dr. Chernov — Final Exam

8 problems. Show your work. — April 2013

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Define $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi(x,y) = \big(x + f(x+y), y - f(x+y)\big)$. Prove that $\mathbf{m}(\Phi(E)) = \mathbf{m}(E)$ for each measurable set $E \subset \mathbb{R}^2$.

2. Let μ and ν be finite positive measures on a measurable space $(X, \mathfrak{M})$ such that $\mu \ll \nu$ and $\nu \ll \mu$. Show that $\nu \ll \mu + \nu$ and the Radon-Nikodym derivative of ν with respect to $\mu + \nu$ satisfies

$$0 < \frac{d\nu}{d(\mu+\nu)} < 1$$

a.e. with respect to μ. Is the same true a.e. with respect to $(\mu + \nu)$?

3. Let $E \subset \mathbb{R}$ be a measurable set with positive Lebesgue measure. We say that $x \in \mathbb{R}$ is a *point of positive measure* with respect to E if $\mathbf{m}(E \cap I) > 0$ for each open interval I containing x. Let

$$E^+ = \{x \in \mathbb{R}\colon x \text{ is of positive measure with respect to } E\}$$

(i) Prove that $\mathbf{m}(E \setminus E^+) = 0$.
(ii) Is it always true that $\mathbf{m}(E^+ \setminus E) = 0$?

4. Let $f\colon \mathbb{R} \to \mathbb{C}$ be continuous with compact support, and let $g \in L^1(\mathbb{R})$. Prove that the convolution function

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, d\mathbf{m}$$

is defined and is continuous at every point $x \in \mathbb{R}$.

5. Prove or disprove: the function

$$f(x) = \begin{cases} \frac{1}{\ln x} & x > 0 \\ 0 & x = 0 \end{cases}$$

is absolutely continuous on $[0, \frac{1}{2}]$.

6. Let $f_1\colon [0, M] \to \mathbb{R}$ be a bounded measurable function, i.e., $|f(x)| \le C$ for all $x \in [0, M]$. Define

$$f_{n+1}(x) = \int_{[0,x]} f_n\, d\mathbf{m}$$

for $n = 2, 3, \ldots$. Prove that the series $S(x) = \sum_{n=2}^{\infty} f_n(x)$ is uniformly convergent on $[0, M]$ and the sum is a continuous function on $[0, M]$. Can you give an upper bound for $S(x)$?

7. Let f be a real valued and increasing function on the real line $\mathbb{R}$, such that $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$. Prove that f is absolutely continuous on every closed finite interval $[a,b] \subset \mathbb{R}$ if and only if $\int_{\mathbb{R}} f' \, d\mathbf{m} = 1$.

8. Let $E = [0,1] \times [0,1]$ and
$$f(x,y) = \frac{xy}{(x^2+y^2)^2}$$
for $0 < x, y \leq 1$ and $f(x,y) = 0$ otherwise. Show that $f \notin L^1(E)$.