MA 660-1D (Numerical Linear Algebra), Dr. Chernov — Midterm test

Justify your work. — Mon, March 1, 2004

1. (15 pts) Let $A \in \mathbb{C}^{m\times n}$. Denote by $\lambda_{\max}$ the largest eigenvalue of the positive semidefinite matrix A^*A. Prove that

$$\|Ax\|_2 = \|A\|_2\|x\|_2 \qquad \Longleftrightarrow \qquad A^*Ax = \lambda_{\max}x.$$

2. (10 pts) Prove that the matrix

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

is **not** positive-definite in two ways: (a) by using Sylverster's theorem and (b) by applying Cholesky factorization. Is this matrix positive semidefinite?

3. (10 pts) Prove that if a matrix $A \in \mathbb{R}^{n\times n}$ is orthogonal and upper-triangular, then it is diagonal. What can you say about its diagonal elements?

4. (10 pts) Prove that every real orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ is a product of Householder reflectors. (Hint: use QR decomposition.)

5. (5 pts) Under what condition does a matrix $A \in \mathbb{R}^{n\times n}$ have a unique LU decomposition $A = LU$, where L is unitary lower triangular and U is upper triangular? Show that the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has no LU decomposition, even if we only require that L be lower triangular (not necessarily unit lower triangular). Show that the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

admits multiple LU decompositions, i.e. $A = L_1U_1 = L_2U_2$, where L_1 and L_2 are two *distinct* unit lower triangular matrices and U_1 and U_2 are two *distinct* upper triangular matrices. How many distinct LU decomposition does it admit?