MA 660-3A (Numerical Linear Algebra), Dr. Chernov — Midterm test

Justify your work. — Mon, March 17, 2008

1. Let $A \in \mathbb{C}^{n\times n}$ be unitary, Hermitean and positive definite. Show that $A = I$.

Solution: by the spectral theorem, $A = Q^* D Q$, where Q is a unitary matrix and D is a diagonal matrix; the eigenvalues of A are on the diagonal of D. Now the properties of A imply that all the eigenvalues of A are equal to 1, hence $D = I$, so $A = Q^*Q = I$.

2. Use a QR decomposition, with exact arithmetic, to solve the least squares problem for the overdetermined system

$$\begin{bmatrix} 2 & 1 \\ -2 & 8 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -11 \\ -5 \end{bmatrix}$$

Solution: the (short) QR decomposition of the matrix of coefficients is

$$\begin{bmatrix} 2 & 1 \\ -2 & 8 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ -2/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 0 & 9 \end{bmatrix},$$

and $x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

3. (JPE, September 1996) Compute the singular values of the matrix

$$A = \begin{bmatrix} 0 & -1.6 & 0.6 \\ 0 & 1.2 & 0.8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: first we compute

$$A^*A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so the eigenvalues of A^*A are 4, 1, 0 (in decreasing order). The singular values of A are 2, 1, 0 (they must be listed in decreasing order).

4. Let $A = QTQ^*$ be a Schur decomposition of the matrix

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

Find the matrix T.

Solution: first we check that $A^*A = AA^*$, hence A is normal. Thus in its Shur decomposition, the matrix T is diagonal. The diagonal entries of T are the eigenvalues of A. They are $2i$ and $-2i$. (They may be ordered arbitrarily, so there are two possible matrices T).

5. Prove that the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}.$$

is positive definite in two ways. First, use Sylvester's theorem. Second, find the Cholesky factorization for A.

Solution: by Sylvester's theorem, $\det A_1 = 1 > 0$ and $\det A_2 = 8 - 4 = 4 > 0$, hence A is positive definite. Its Cholesky factorization is

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}.$$

Bonus. Let A be a 3×3 symmetric positive definite real matrix. The Cholesky factorization says that there is a unique lower triangular matrix G with positive diagonal elements such that $A = GG^T$. How many lower triangular real matrices G_i exist that satisfy the equation $A = G_iG_i^T$? [Hint: use the uniqueness of LDL^T-decomposition.]

Answer: $2^3 = 8$. The idea is that $G_i = LD^{1/2}$, and the square roots of the (three) diagonal entries of D may assume different signs.