MA 660-3A (Numerical Linear Algebra), Dr. Chernov — Midterm test

Take home test. Justify your work. — March 2010

1. Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Is there any relationship between the singular values of A and the eigenvalues of A? If yes, describe it and prove it.

2. Let $A \in \mathbb{R}^{n\times n}$ have singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Find all the singular values of the $2n \times 2n$ symmetric matrix

$$B = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}.$$

What are the eigenvalues of B?

3. Construct a Schur decomposition for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

4. Consider least squares problem $\min\limits_x \|Ax - b\|_2$ with $A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

(a) Solve it by using normal equations.

(b) Compute the reduced singular value decomposition of A, then use it to solve the above least squares problem.

5. Let $A = I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$, where I is an $n \times n$ identity matrix and $\mathbf{1}$ is a column vector of length n all of whose entries are equal to 1, i.e. $\mathbf{1} = [\underbrace{1, \ldots, 1}_{n}]^T$.

Prove that the singular values of A are $\underbrace{1, 1, \ldots, 1}_{n-1}, 0$.

6. Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. The corresponding *dual norm* $\|\cdot\|'$ is defined by the formula $\|x\|' = \sup_{\|y\|=1} |y^* x|$.

(a) Prove that the $\|\cdot\|_1$ and $\|\cdot\|_\infty$ norms are dual to each other.

(b) Prove that $\|\cdot\|$ coincides with $\|\cdot\|'$ if and only if it is the 2-norm.