MA 660-1D (Numerical Linear Algebra), Dr. Chernov — Final exam
Justify your work. — Fri, Apr 30, 2004

1. Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Is there any relationship between the singular values of A and the eigenvalues of A?

2. Let $A \in \mathbb{C}^{n\times n}$ be unitary. What are the singular values of A? How would you write an SVD for A?

3. Let $A \in \mathbb{R}^{n\times n}$ have singular values $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_n$. Find all the singular values of the $2n \times 2n$ symmetric matrix

$$B = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}.$$

What are the eigenvalues of B?

4. Let $A \in \mathbb{C}^{n\times n}$ be an upper Hessenberg matrix with all its sub-diagonal entries nonzero (that is, $a_{i+1,i} \neq 0$ for all $1 \leq i \leq n-1$). Prove that for every $\lambda \in \mathbb{C}$ the matrix $A - \lambda I$ has rank at least $n-1$. Then prove that each eigenvalue of A has geometric multiplicity one.

5. Find the condition numbers of the eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & -6 \\ 4 & -10 \end{bmatrix}$$

6. Suppose $A \in \mathbb{C}^{n\times n}$ and $A^2 = I$. What can you say about the minimal polynomial of A? What does this imply about the Jordan decomposition of A?

7. Construct a Schur decomposition for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$