MA 660-3A (Numerical Linear Algebra), Dr. Chernov — Final Exam
Justify your work. — April 2010

1. With the initial vector $q_0 = (1,1)^T$, calculate two iterations (i.e., q_1 and q_2, along with $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$) of the power method with Rayleigh quotient for the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}.$$

The choice of the scaling factor σ_k is yours.

2. Compute the condition numbers κ_1, κ_2 and κ_∞ for the matrix

$$A = \begin{pmatrix} 2 & 4 \\ 1.01 & 2 \end{pmatrix}$$

3. Find the condition number of each eigenvalue of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

4. Use a Givens rotator to compute a QR decomposition for the matrix

$$A = \begin{bmatrix} 4 & 5 & -10 \\ 3 & 10 & 5 \\ 0 & 0 & 3 \end{bmatrix}.$$

5. Apply the QR algorithm (without shift) to the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Does it converge and produce the eigenvalues of A? If not, why? Apply the QR algorithm with the Rayleigh quotient shift. Does it help the convergence? Why? Lastly, try the QR algorithm with the Wilkinson shift. Does it do the trick?

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6. Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and $\lambda_{\max}$ be the largest eigenvalue of A, i.e., $\lambda_{\max} = \max\{\lambda_1, \ldots, \lambda_n\}$. Show that $\lambda_{\max} \geq \max_i a_{ii}$. Hint: use the properties of the Rayleigh quotient.

7. Let $A \in \mathbb{C}^{n\times n}$ be an upper Hessenberg matrix with all its sub-diagonal entries nonzero (i.e., $a_{i+1,i} \neq 0$ for all $1 \leq i \leq n-1$). Prove that for every $\lambda \in \mathbb{C}$ the matrix $A - \lambda I$ has rank at least $n-1$. Then prove that each eigenvalue of A has geometric multiplicity one.

8. Let $n \geq 2$ and $A \in \mathbb{R}^{n\times n}$ be defined by

$$a_{ij} = 9^{-|i-j|} \qquad \text{for} \quad i,j = 1,\ldots,n$$

Show that all the eigenvalues of A lie in the interval $(0.75, 1.25)$. Hint: use the Gershgorin theorem.