MA 660-1F, Dr Chernov | Assignment #1
. | Due Mon, Jan 23

1. Show that the norm $\|A\| = \max_{i,j} |a_{ij}|$ on the space of $n \times n$ real matrices is not induced by any vector norm.

2. Prove the Neuman lemma: if $\|A\| < 1$, then $I - A$ is invertible. Here $\|\cdot\|$ is a matrix norm induced by a vector norm.

3. Let V be an inner product space, and $\|\cdot\|$ denote the norm induced by the inner product. Prove the **parallelogram law**

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

Based on this, show that the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ in $\mathbb{C}^n$ are not induced by any inner products.

4. Let $\{u_1, \dots, u_n\}$ be an ONB in $\mathbb{C}^n$. Assuming that n is even, compute

$$\|u_1 - u_2 + u_3 - \cdots - u_n\|$$

5. Let V be an inner product space and $W \subset V$ a finite dimensional subspace with an ONB $\{u_1, \dots, u_n\}$. For every $x \in V$ define

$$P(x) = \sum_{i=1}^{n} \langle x, u_i \rangle u_i.$$

(i) Prove that $x - P(x) \in W^\perp$, hence P is the orthogonal projection onto W.
(ii) Prove that $\|x - P(x)\| \le \|x - z\|$ for every $z \in W$, and that if $\|x - P(x)\| = \|x - z\|$ for some $z \in W$, then $z = P(x)$.

6. (JPE, May 1999). Let $P \in \mathbb{C}^{n\times n}$ be a projector. Show that $\|P\|_2 \ge 1$ with equality if and only if P is an orthogonal projector.

7. Let $A \in \mathbb{C}^{m\times n}$. Show that

$$\|UA\|_2 = \|AV\|_2 = \|A\|_2$$

for any unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$.

Continued on back

8. Let $A \in \mathbb{C}^{m\times n}$ and $\|A\|_F^2 = \sum_{i,j} |a_{ij}|^2$ (Frobenius norm). Show that

$$\|UA\|_F = \|AV\|_F = \|A\|_F$$

for any unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$.

9. Show that if Q is a real orthogonal 2×2 matrix and $\det Q = 1$, then

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some $\theta \in [0, 2\pi)$ (i.e. Q represents a rotation of $\mathbb{R}^2$).

10. Show that if Q be a real orthogonal 2×2 matrix and $\det Q = -1$, then

$$Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

for some $\theta \in [0, 2\pi)$. Then prove that $\lambda = \pm 1$ are eigenvalues of the above matrix Q. (This means that Q reflects $\mathbb{R}^2$ across a line.)

11. Show that if a matrix A is both triangular and unitary, then it is diagonal.