MA 660-3A, Dr Chernov — Assignment #13

. — Due Thu, Apr 24

1. (JPE, May 1989) Let A be a real symmetric matrix, with eigenvalues λ_i, $1 \le i \le n$, satisfying

$$|\lambda_1| > |\lambda_2| \ge \cdots \ge |\lambda_n|$$

If x_1 is an eigenvector corresponding to λ_1, and z_0 is a vector satisfying $z_0^t x_1 \ne 0$, prove that

$$\lim_{k\to\infty} \frac{z_0^t A^k z_0}{z_0^t A^{k-1} z_0} = \lambda_1$$

2. (JPE, May 2003) Let A be a symmetric matrix with eigenvalues such that $|\lambda_1| > |\lambda_2| \ge \cdots |\lambda_{n-1}| > |\lambda_n|$. Suppose $z \in \mathbb{R}^n$ with $z^T x_1 \ne 0$, where $Ax_1 = \lambda_1 x_1$. Prove that, for some constant C,

$$\lim_{k\to\infty} \frac{A^k z}{\lambda_1^k} = Cx_1$$

and use this result to devise a reliable algorithm for computing λ_1 and x_1. Explain how the calculation should be modified to obtain (a) λ_n and (b) the eigenvalue closest to 2.

Bonus. (JPE, September 1996) The matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

has eigenpairs

$$(\lambda, x) = \left(2, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \left(-1, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right), \left(3, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right),$$

Suppose the power method is applied with starting vector

$$z_0 = [1, 1, -1]^t/\sqrt{3}$$

(a) Determine whether or not the iteration will converge to an eigenpair of A, and if so, which one. Assume exact arithmetic.
(b) Repeat (a), except now use the inverse iteration with the same starting vector z_0 and the Rayleigh quotient of z_0 as approximation for the eigenvalue.
(c) Now answer both (a) and (b) again, except this time use standard fixed precision floating point arithmetic, i.e. computer arithmetic.