MA 660-2C, Dr Chernov — Warm-up homework assignment #2

. Due Tue, Sep 4

1. Problem 2.1 from Textbook.

2. Let $\{w_1, \dots, w_n\}$ be an ONB in $\mathbb{R}^n$. Assuming that n is even, compute

$$\|w_1 - w_2 + w_3 - \cdots - w_n\|$$

3. Apply Gram-Schmidt orthonormalization to the the basis $\{1, x, x^2\}$ in the space $P_2(\mathbb{R})$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$$

4. (i) Let $W \subset V$ be a subspace of an inner product space V. Prove that $W \subset (W^\perp)^\perp$.
(ii) If, in addition, V is finite dimensional, prove that $W = (W^\perp)^\perp$.

5. Let V be an inner product space and $W \subset V$ a finite dimensional subspace with ONB $\{u_1, \dots, u_n\}$. For every $x \in V$ define $P(x) = \sum_{i=1}^n \langle x, u_i \rangle u_i$.
(i) Prove that $x - P(x) \in W^\perp$
(ii) Prove that P is the orthogonal projection on W
(iii) Prove that $||x - P(x)|| \le ||x - z||$ for every $z \in W$, and that if $||x - P(x)|| = ||x - z||$ for some $z \in W$, then $z = P(x)$.

6. (i) Show that if a, b are real numbers satisfying $a^2 + b^2 = 1$, then $a = \cos\theta$ and $b = \sin\theta$ for some $\theta \in [0, 2\pi)$.
(ii) Show that if Q is a real orthogonal 2×2 matrix and $\det Q = 1$, then $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in [0, 2\pi)$ (i.e. Q represents a rotation of $\mathbb{R}^2$).

7. [Extra credit] Let $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ for some $\theta \in [0, 2\pi)$. Show that A is orthogonal and $\det A = -1$. Prove that $\lambda = \pm 1$ are eigenvalues of the above matrix A and find its eigenvectors. Describe the action of A in $\mathbb{R}^2$ geometrically. Show that if A is a 2×2 orthogonal matrix and $\det A = -1$, then A has the above form.