MA 660-1D, Dr Chernov

Homework assignment #2

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Due Mon, Jan 26

1. (i) Let $W \subset V$ be a subspace of an inner product space V. Prove that $W \subset (W^\perp)^\perp$.
(ii) If, in addition, V is finite dimensional, prove that $W = (W^\perp)^\perp$.

2. Let V be an inner product space and $W \subset V$ a finite dimensional subspace with ONB $\{u_1, \dots, u_n\}$. For every $x \in V$ define $P(x) = \sum_{i=1}^n \langle x, u_i \rangle u_i$.
(i) Prove that $x - P(x) \in W^\perp$
(ii) Prove that P is the orthogonal projection on W
(iii) Prove that $||x - P(x)|| \le ||x - z||$ for every $z \in W$, and that if $||x - P(x)|| = ||x - z||$ for some $z \in W$, then $z = P(x)$.

3. For an $m \times n$ complex matrix A define $||A||_2 := \sup_{||x||_2=1} ||Ax||_2$, where $||x||_2$ is the 2-norm in $\mathbb{C}^n$ and $||Ax||_2$ is the 2-norm in $\mathbb{C}^m$.Prove that for every vector $z \in \mathbb{C}^n$ we have $||z||_2 = \sup_{||y||_2=1} |\langle y, z \rangle|$, and then show that $||A^*A||_2 = ||A||_2^2 = ||A^*||_2^2$.

4. The spectral radius of a real $n \times n$ matrix A is defined by

$$\rho(A) := \max\{|\lambda| : \ \lambda \text{ eigenvalue of } A\}$$

Show that
(a) $\rho(A) \le ||A||$ for every matrix norm $|| \cdot ||$ that is induced by a norm on $\mathbb{R}^n$
(b) if A is symmetric then $||A||_2 = \rho(A)$ (Hint: use the spectral theorem)
(c) $||A||_2^2 = \rho(A^t A)$

5. Show that if Q is a real orthogonal 2×2 matrix and $\det Q = 1$, then $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in [0, 2\pi)$ (i.e. Q represents a rotation of $\mathbb{R}^2$).

6. Show that if Q be a real orthogonal 2×2 matrix and $\det Q = -1$, then $Q = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ for some $\theta \in [0, 2\pi)$. Then prove that $\lambda = \pm 1$ are eigenvalues of the above matrix Q and find its eigenvectors. Describe the action of Q on $\mathbb{R}^2$ geometrically.

[Extra credit] Let AQ be a 3×3 real orthogonal matrix with $\det Q = 1$. Prove that $\lambda = 1$ is an eigenvalue of Q. [Hint: use 10.12 and the previous problem]