MA 660-3A, Dr Chernov

Assignment #2

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Due Wed, Jan 23

1. Apply Gram-Schmidt orthonormalization to the the basis $\{1, x, x^2\}$ in the space $\mathbb{P}_2(\mathbb{R})$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$$

2. (i) Let $W \subset V$ be a subspace of an inner product space V. Prove that $W \subset (W^\perp)^\perp$.
(ii) If, in addition, V is finite dimensional, prove that $W = (W^\perp)^\perp$.

3. Let V be an inner product space and $W \subset V$ a finite dimensional subspace with ONB $\{u_1, \dots, u_n\}$. For every $x \in V$ define $P(x) = \sum_{i=1}^n \langle x, u_i\rangle u_i$.
(i) Prove that $x - P(x) \in W^\perp$, hence P is the orthogonal projection onto W.
(ii) Prove that $\|x - P(x)\| \le \|x - z\|$ for every $z \in W$, and that if $\|x - P(x)\| = \|x - z\|$ for some $z \in W$, then $z = P(x)$.

4. Show that if Q is a real orthogonal 2×2 matrix and $\det Q = 1$, then $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in [0, 2\pi)$ (i.e. Q represents a rotation of $\mathbb{R}^2$).

5. Show that if Q be a real orthogonal 2×2 matrix and $\det Q = -1$, then $Q = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ for some $\theta \in [0, 2\pi)$. Then prove that $\lambda = \pm 1$ are eigenvalues of the above matrix Q. Show that Q is a reflection of $\mathbb{R}^2$ across a line.

6. Let $A \in \mathbb{C}^{m\times n}$. Show that

$$\|UA\|_2 = \|AV\|_2 = \|A\|_2$$

for any unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$.