MA 660-1D, Dr Chernov

Homework assignment #3

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Due Mon, Feb 2

1. Let $A = (a_{ij})$ be a complex $n \times n$ matrix. Assume that $\langle Ax, x\rangle = 0$ for all $x \in \mathbb{C}^n$. Prove that
(a) $a_{ii} = 0$ for $1 \le i \le n$ by substituting $x = e_i$
(b) $a_{ij} = 0$ for $i \ne j$ by substituting $x = pe_i + qe_j$ then using (a) and putting $p, q = \pm 1, \pm i$ (here $i = \sqrt{-1}$) in various combinations
Conclude that $A = 0$.

2. Find a real $n \times n$ matrix $A \ne 0$ such that $\langle Ax, x\rangle = 0$ for all $x \in \mathbb{R}^n$.

3. Find a real $n \times n$ matrix A such that $\langle Ax, x\rangle > 0$ for all $x \ne 0$, but A is not symmetric. Hence, the symmetry requirement in Definition 12.9 cannot be dropped in the real case.

4. (JPE, May 1994) Let $A \in \mathbb{R}^{n\times n}$ be given, symmetric and positive definite. Define $A_0 = A$, and consider the sequence of matrices defined by

$$A_k = G_k G_k^t \qquad \text{and} \qquad A_{k+1} = G_k^t G_k$$

where $A_k = G_k G_k^t$ is the Cholesky factorization for A_k. Prove that the A_k all have the same eigenvalues.

5. Let $A \in \mathbb{C}^{n\times n}$ and J a Jordan canonical form of A. Show that A has a square root (in the complex sense!) if and only if so does J. Show that if J is diagonal, then both J and A have square roots.

[Extra credit] Let $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ be a nondiagonal Jordan block. Show that J has a square root if and only if $\lambda \ne 0$.