MA 660-3A, Dr Chernov | Assignment #3
. | Due Mon, February 1

1. The spectral radius of $A \in \mathbb{C}^{n\times n}$ is defined by

$$\rho(A) = \max\{|\lambda| \colon \lambda \text{ is an eigenvalue of } A\}$$

Show that $\rho(A) \le \|A\|$ for every matrix norm $\|\cdot\|$ that is induced by a vector norm.

2. Let $x, y \in \mathbb{C}^n$ and $A = yx^*$. Show that $\operatorname{rank} A = 1$. Show that $\|A\|_2 = \|x\|_2\|y\|_2$. Show that $\|A\|_F = \|x\|_F\|y\|_F$, where $\|\cdot\|_F$ stands for the Frobenius norm.

3. Let $A \in \mathbb{C}^{m\times n}$. Show that

$$\|UA\|_F = \|AV\|_F = \|A\|_F$$

for any unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$. Here again $\|\cdot\|_F$ stands for the Frobenius norm.

4 (Bonus) Let $A = (a_{ij})$ be a complex $n \times n$ matrix. Assume that $\langle Ax, x\rangle = 0$ for all $x \in \mathbb{C}^n$. Prove that
(a) $a_{ii} = 0$ for $1 \le i \le n$ by substituting $x = e_i$
(b) $a_{ij} = 0$ for $i \ne j$ by substituting $x = pe_i + qe_j$ then using (a) and putting $p, q = \pm 1, \pm\mathbf{i}$ (here $\mathbf{i} = \sqrt{-1}$) in various combinations.
Conclude that $A = 0$.