MA 660-3A, Dr Chernov — Assignment #4

. — Due Mon, Feb 4

1. Prove that for every vector $z \in \mathbb{C}^n$ we have

$$\|z\|_2 = \sup_{\|y\|_2=1} |\langle y, z\rangle|.$$

Then show that for every $A \in \mathbb{C}^{m\times n}$

$$\|A^*A\|_2 = \|A\|_2^2 = \|A^*\|_2^2 = \|AA^*\|_2.$$

2. The spectral radius of $A \in \mathbb{C}^{n\times n}$ is defined by

$$\rho(A) = \max\{|\lambda| : \ \lambda \text{ eigenvalue of } A\}.$$

Show that
(a) $\rho(A) \le \|A\|$ for every matrix norm $\|\cdot\|$ that is induced by a vector norm.
(b) if A is Hermitian, then $\|A\|_2 = \rho(A)$ (Hint: use the spectral theorem)
(c) $\|A\|_2^2 = \rho(A^*A) = \rho(AA^*)$.

3. Let $A \in \mathbb{C}^{m\times n}$. Prove that

$$\|Ax\|_2 = \|A\|_2\|x\|_2 \qquad \Longleftrightarrow \qquad A^*Ax = \lambda_{\max}x.$$

where $\lambda_{\max}$ is the largest eigenvalue of both A^*A and AA^*.

4. Let $x, y \in \mathbb{C}^n$ and $A = yx^*$. Show that $\operatorname{rank} A = 1$. Show that $\|A\|_2 = \|x\|_2\|y\|_2$. Show that $\|A\|_F = \|x\|_F\|y\|_F$, where $\|\cdot\|_F$ stands for the Frobenius norm.

Bonus (JPE, May 1999). Let $P \in \mathbb{C}^{n\times n}$ be a projector. Show that $\|P\|_2 \ge 1$ with equality if and only if P is an orthogonal projector.