

1. Find the numerical rank with tolerance 0.9 of the matrix

$$A = \begin{pmatrix} 3 & 2 \\ -4 & -5 \end{pmatrix}$$

2. (combined from JPE, October 1990 and May 1997) Let $A \in \mathbb{C}^{n \times n}$ be a normal matrix.

(a) Prove that $A - \lambda I$ is normal for any $\lambda \in \mathbb{C}$.

(b) Prove that $\|Ax\| = \|A^*x\|$ for all x .

(c) Prove that (λ, x) is an eigenpair of A if and only if $(\bar{\lambda}, x)$ is an eigenpair of A^* . (Hence, A and A^* have the same eigenvectors.)

3. (JPE, September 2002) A matrix $A \in \mathbb{C}^{n \times n}$ is said to be *skew Hermitian* if $A^* = -A$.

(a) Prove that if A is skew Hermitian and B is unitary equivalent to A , then B is also skew Hermitian.

(b) Prove that the eigenvalues of a skew Hermitian matrix are purely imaginary, i.e. they satisfy $\bar{\lambda} = -\lambda$.

(c) What special form does the Schur decomposition take for a skew Hermitian matrix A ?

[Bonus] (JPE, May 1996). Let T be a linear operator on a finite dimensional complex inner product space V , and let T^* be the adjoint of T . Prove that $T = T^*$ if and only if $T^*T = T^2$. (Hint: reduce the problem to complex matrices and use the Schur decomposition.)

[Bonus] (JPE, September 1998). Show that diagonalizable (complex) matrices make a dense subset of $\mathbb{C}^{n \times n}$. That is, for any $A \in \mathbb{C}^{n \times n}$ and $\varepsilon > 0$ there is a diagonalizable $B \in \mathbb{C}^{n \times n}$ such that $\|A - B\|_2 < \varepsilon$. (Hint: use Schur decomposition theorem).