MA 660-3A, Dr Chernov

Assignment #5

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Due Mon, Feb 15

1. (combined from JPE, October 1990 and May 1997) Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix.
(a) Prove that $A - \lambda I$ is normal for any $\lambda \in \mathbb{C}$.
(b) Prove that $\|Ax\|_2 = \|A^*x\|_2$ for all x.
(c) Prove that (λ, x) is an eigenpair of A if and only if $(\bar{\lambda}, x)$ is an eigenpair of A^*. (Hence, A and A^* have the same eigenvectors.)

2. (JPE, May 1996). Let T be a linear operator on a finite dimensional complex inner product space V, and let T^* be the adjoint of T. Prove that $T = T^*$ if and only if $T^*T = T^2$. (Hint: reduce the problem to complex matrices and use the Schur decomposition.)

3. (JPE, September 1998). Show that diagonalizable (complex) matrices make a dense subset of $\mathbb{C}^{n\times n}$. That is, for any $A \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$ there is a diagonalizable $B \in \mathbb{C}^{n\times n}$ such that $\|A - B\|_2 < \varepsilon$. (Hint: use Schur decomposition theorem).

4. (JPE, September 2002) A matrix $A \in \mathbb{C}^{n\times n}$ is said to be *skew Hermitian* if $A^* = -A$.
(a) Prove that if A is skew Hermitian and B is unitary equivalent to A, then B is also skew Hermitian.
(b) What special form does the Schur decomposition take for a skew Hermitian matrix A?
(c) Prove that the eigenvalues of a skew Hermitian matrix are purely imaginary, i.e. they satisfy $\bar{\lambda} = -\lambda$.

Bonus. Prove that for any $A \in \mathbb{C}^{n\times n}$

$$\lim_{n\to\infty} \|A^n\| = 0 \quad \Longleftrightarrow \quad \rho(A) < 1$$

where $\|\cdot\|$ is any matrix norm induced by a vector norm. Hint: use Schur decomposition.