MA 660-1F, Dr Chernov Assignment #5
. Due Mon, Feb 20

1. (JPE, May 1994) Let $A \in \mathbb{R}^{n\times n}$ be given, symmetric and positive definite. Define $A_0 = A$, and consider the sequence of matrices defined by

$$A_k = G_k G_k^t \qquad \text{and} \qquad A_{k+1} = G_k^t G_k$$

where $A_k = G_k G_k^t$ is the Cholesky factorization for A_k. Prove that the A_k all have the same eigenvalues.

2. Find a nonzero matrix $A \in \mathbb{R}^{2\times 2}$ that admits at least two LU decomposition, i.e. $A = L_1U_1 = L_2U_2$, where L_1 and L_2 are two *distinct* unit lower triangular matrices and U_1 and U_2 are two *distinct* upper triangular matrices.

3. Show that the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ admits no LU decomposition, even if we only require that L be lower triangular (not necessarily unit lower triangular).

4. The spectral radius of a matrix $A \in \mathbb{C}^{n\times n}$ is defined by

$$\rho(A) = \max\{|\lambda| : \ \lambda \text{ eigenvalue of } A\}.$$

(a) Show that $\rho(A) \le \|A\|_2$.
(b) Give an example of a 2×2 matrix A such that $\rho(A) < 1$ but $\|A\|_2 > 100$.
(c) Show that if

$$\lim_{n\to\infty} \|A^n\|_2 = 0,$$

then $\rho(A) < 1$.

[Bonus] In the notation of the previous problem, show that if $\rho(A) < 1$, then

$$\lim_{n\to\infty} \|A^n\|_2 = 0.$$

Hint: use Jordan decomposition.