

1. (JPE, May 1994) Let $A \in \mathbb{R}^{n \times n}$ be given, symmetric and positive definite. Define $A_0 = A$, and consider the sequence of matrices defined by

$$A_k = G_k G_k^T \quad \text{and} \quad A_{k+1} = G_k^T G_k$$

where $A_k = G_k G_k^T$ is the Cholesky factorization for A_k . Prove that the A_k all have the same eigenvalues.

2. (JPE, September 2002) Consider three vectors

$$v_1 = \begin{pmatrix} 1 \\ \epsilon \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ \epsilon \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \epsilon \end{pmatrix}.$$

where $\epsilon \ll 1$.

(a) Use the classical Gram-Schmidt method to compute 3 **orthonormal** vectors q_1, q_2, q_3 , making the approximation that $1 + \epsilon^2 \approx 1$ (that is, replace any term containing ϵ^2 or smaller with zero, **but** retain terms containing ϵ). Are q_i ($i = 1, 2, 3$) pairwise orthogonal? If not, why not?

(b) Repeat (a) using the modified Gram-Schmidt orthogonalization process. Are the q_i ($i = 1, 2, 3$) pairwise orthogonal? If not, why not?